BOARD OF STUDIES
NEW SOUTH WALES

2002

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate, correct to three significant figures, **2**

$$\frac{5.8^2 - 3.1^3}{3 \times 3.1 \times 5.8}.$$

(b) Differentiate $x^3 + 2$. **2**

(c) Solve $x^2 = 5x$. **2**

(d) Integrate $\frac{3}{x}$. **1**

(e) Solve $3x - \frac{2x-5}{2} = 6$. **3**

(f) Solve the pair of simultaneous equations **2**

$$x - 2y = 8$$
$$2x + y = 1.$$

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

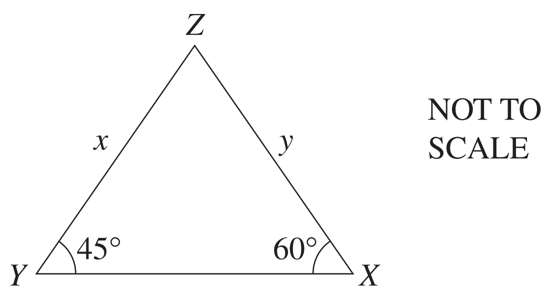
(a) Find the equation of the tangent to $y = e^{2x}$ at the point (0, 1). **2**

(b) Differentiate:

(i) $x \sin x$ **2**

(ii) $\dfrac{\ln x}{x^2}$. **2**

(c) **3**



In the diagram, XYZ is a triangle where $\angle ZYX = 45°$ and $\angle ZXY = 60°$.

Find the exact value for the ratio $\dfrac{x}{y}$.

(d) Find:

(i) $\int \cos 3x \, dx$ **1**

(ii) $\int_0^1 \left(e^{5x} - 1\right) dx$. **2**

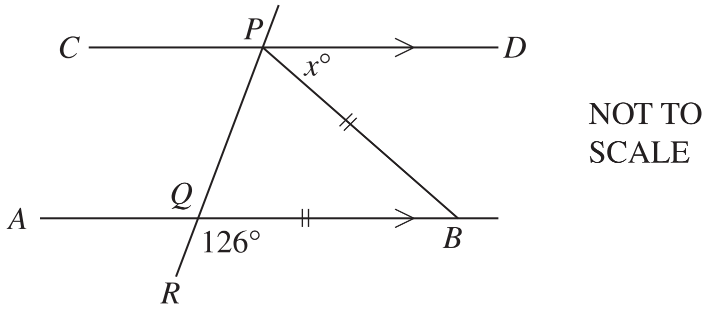
Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Josh invests \$1000 in a term deposit that earns 3.5% per annum compounded annually. **2**

What is the value of the investment at the end of 20 years?

(b) **3**

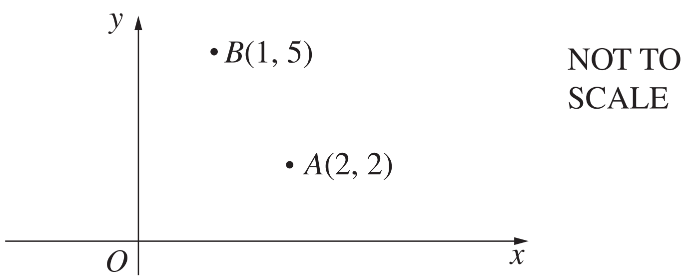


In the diagram, CD is parallel to AB, $PB = QB$, $\angle BQR = 126°$ and $\angle BPD = x°$.

Copy or trace this diagram into your writing booklet.

Find the value of x, giving complete reasons.

(c)



The diagram shows two points $A(2, 2)$ and $B(1, 5)$ on the number plane.

Copy the diagram into your writing booklet.

(i) Find the coordinates of M, the midpoint of AB. **1**

(ii) Show that the equation of the perpendicular bisector of AB is $x - 3y + 9 = 0$. **2**

(iii) Find the coordinates of the point C that lies on the y axis and is equidistant from A and B. **1**

(iv) The point D lies on the intersection of the line $y = 5$ and the perpendicular bisector $x - 3y + 9 = 0$. Find the coordinates of D, and mark the position of D on your diagram in your writing booklet. **1**

(v) Find the area of triangle ABD. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

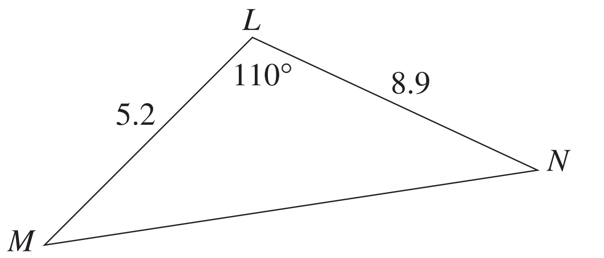
(a) Solve $|x-1| \geq 3$ and graph your solution on the number line. **2**

(b) Find all values of θ, where $0° \leq \theta \leq 360°$, that satisfy the equation **2**

$$\cos\theta - \frac{2}{5} = 0 \ .$$

Give your answer(s) to the nearest degree.

(c)



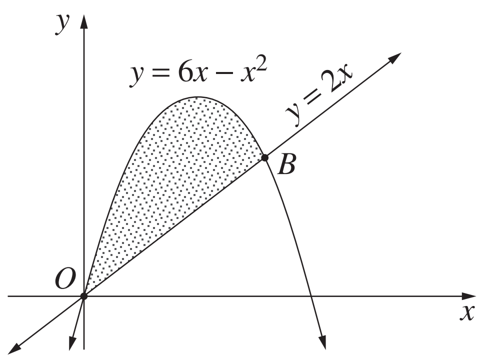
NOT TO SCALE

In the diagram, LMN is a triangle where $LM = 5.2$ metres, $LN = 8.9$ metres and angle $MLN = 110°$.

(i) Find the length of MN. **2**

(ii) Calculate the area of triangle LMN. **2**

(d)



NOT TO SCALE

The graphs of $y = 2x$ and $y = 6x - x^2$ intersect at the origin and point B.

(i) Show that the coordinates of B are (4, 8). **1**

(ii) Find the shaded area bounded by $y = 6x - x^2$ and $y = 2x$. **3**

Marks

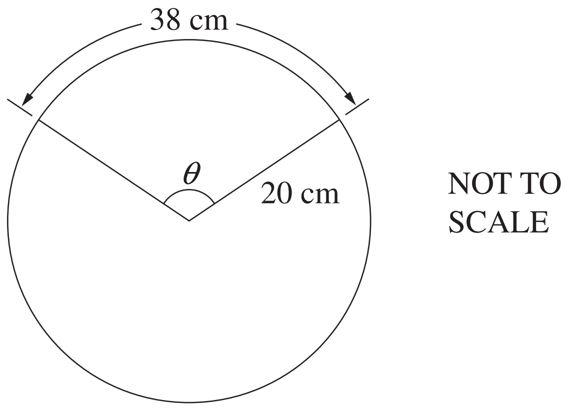
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Catrine is exercising her dog by throwing a stick for the dog to fetch and return. The first time, Catrine throws the stick 2 m and she continues to throw the stick after the dog has returned it. Each time, she increases the distance the stick is thrown by exactly 1.5 m. Her last throw is 32 m.

(i) How many times did Catrine throw the stick? **2**

(ii) How far did her dog run altogether in fetching and returning the stick? (Assume that the dog starts and finishes at Catrine's side.) **3**

(b) **3**



The length of the arc between two spokes on a car's steering wheel is 38 cm. Each spoke is 20 cm in length.

Calculate the angle θ between the two spokes. Give your answer correct to the nearest degree.

(c) Consider the parabola $y = x^2 - 8x + 4$. Find:

(i) the coordinates of the vertex, **2**

(ii) the coordinates of the focus. **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Sketch the graph of $y = \sqrt{4 - x^2}$, and state the range. **2**

(b) The gradient function of a curve is given by

$$f'(x) = 3(x+1)(x-3)$$

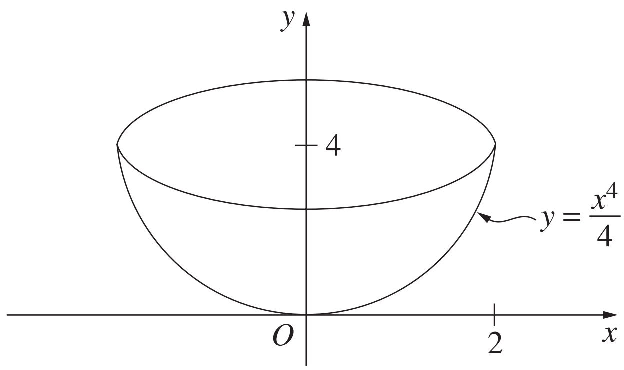
and the curve $y = f(x)$ passes through the point (0, 12).

(i) Find the equation of the curve $y = f(x)$. **2**

(ii) Sketch the curve $y = f(x)$, clearly labelling turning points and the y intercept. **3**

(iii) For what values of x is the curve concave up? **1**

(c) **4**



NOT TO SCALE

A bowl is formed by rotating the part of the curve $y = \dfrac{x^4}{4}$ between $x = 0$ and $x = 2$ about the y axis.

Find the volume of the bowl.

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the geometric series

$$1+\left(\sqrt{5}-2\right)+\left(\sqrt{5}-2\right)^2+\ldots \quad .$$

(i) Explain why the geometric series has a limiting sum. **1**

(ii) Find the exact value of the limiting sum. Write your answer with a rational denominator. **2**

(b) A cooler, which is initially full, is drained so that at time t seconds the volume of water V, in litres, is given by

$$V = 25\left(1-\frac{t}{60}\right)^2 \quad \text{for} \quad 0 \le t \le 60.$$

(i) How much water was initially in the cooler? **1**

(ii) After how many seconds was the cooler one-quarter full? **2**

(iii) At what rate was the water draining out when the cooler was one-quarter full? **2**

(c) Chris has four pairs of socks in a drawer, each pair a different colour.

He selects socks one at a time and at random from the drawer.

(i) The probability that he does NOT have a matching pair after selecting the second sock is $\frac{6}{7}$. Explain why this is so. **1**

(ii) Find the probability that he does NOT have a matching pair after selecting the third sock. **2**

(iii) What is the probability that the first three socks include a matching pair? **1**

Marks

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) A drug is used to control a medical condition. It is known that the quantity Q of drug remaining in the body after t hours satisfies an equation of the form

$$Q = Q_o e^{-kt}$$

where Q_o and k are constants.

The initial dose is 6 milligrams and after 15 hours the amount remaining in the body is half the initial dose.

(i) Find the values of Q_o and k. **3**

(ii) When will one-eighth of the initial dose remain? **2**

(b) A particle moves in a straight line. At time t seconds, its distance x metres from a fixed point O on the line is given by

$$x = \sin 2t + 3.$$

(i) Sketch the graph of x as a function of t for $0 \leq t \leq 2\pi$. **3**

(ii) Using your graph, or otherwise, find the times when the particle is at rest, and the position of the particle at those times. **2**

(iii) Describe the motion completely. **2**

Marks

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the function $y = \ln(x - 1)$ for $x > 1$.

(i) Sketch the function, showing its essential features. **2**

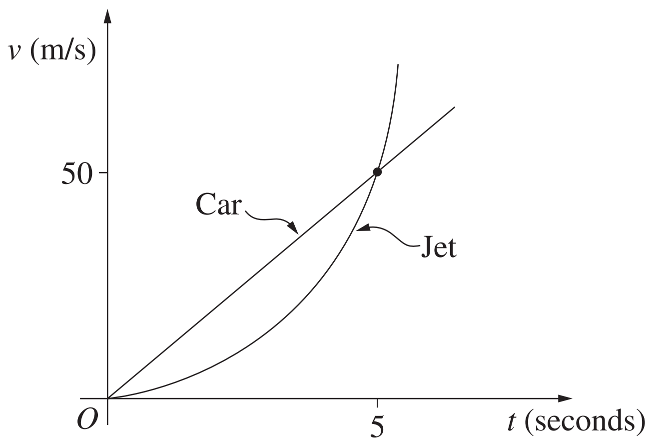
(ii) Use Simpson's rule with three function values to find an approximation to **2**

$$\int_2^4 \ln(x-1)\,dx.$$

(b) A superannuation fund pays an interest rate of 8.75% per annum which compounds annually. Stephanie decides to invest \$5000 in the fund at the beginning of each year, commencing on 1 January 2003. **4**

What will be the value of Stephanie's superannuation when she retires on 31 December 2023?

(c)



NOT TO SCALE

A car and a jet race one another from rest down a runway. The car increases its speed v_1 at a constant rate, while the speed of the jet is given by $v_2 = 2t^2$. After 5 seconds the car and the jet have the same speed of 50 m/s, as shown on the graph.

(i) Find an equation for the speed v_1 of the car in terms of t. **1**

(ii) How far behind the car is the jet after 5 seconds? **2**

(iii) After how many seconds does the jet catch up with the car? **1**

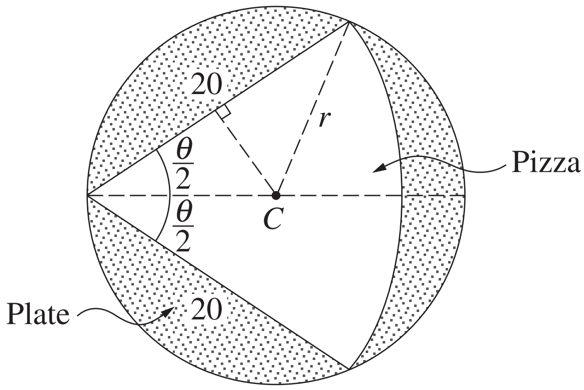
Marks

Question 10 (12 marks) Use a SEPARATE writing booklet.

(a) A circular pizza of radius 20 cm is cut into sectors. Each sector is to be placed on a circular plate that is just large enough to contain that sector.

(i) A sector of pizza is cut where the angle θ at its centre satisfies $0 < \theta \le \frac{\pi}{2}$. **2**

It is placed on a circular plate, of radius r cm and centre C, as shown below.



Show that $r = 10\sec\frac{\theta}{2}$ for $0 < \theta \le \frac{\pi}{2}$.

Question 10 continues on page 13

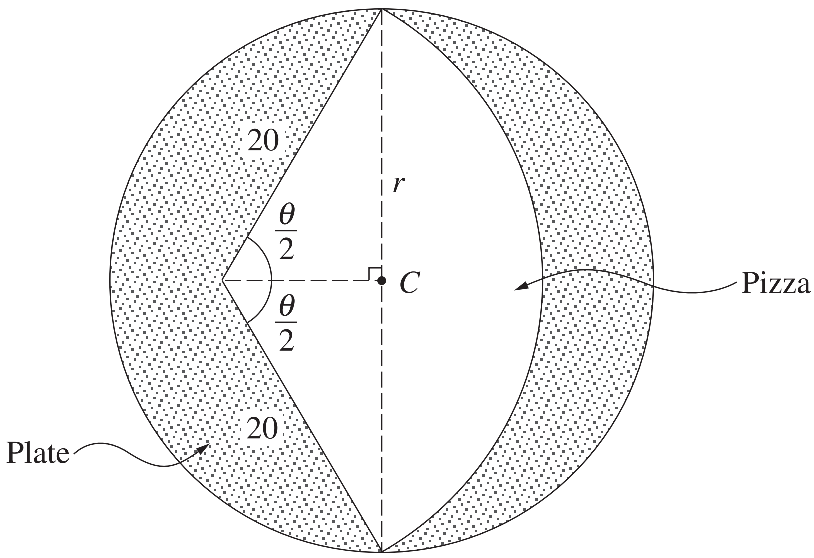
Marks

Question 10 (continued)

(ii) Another sector of pizza is cut where the angle θ at its centre satisfies **1**

$$\frac{\pi}{2} < \theta < \pi .$$

This sector of pizza is placed on a circular plate as shown below. Again, we let the radius of the plate be r cm, and we let the centre be C.



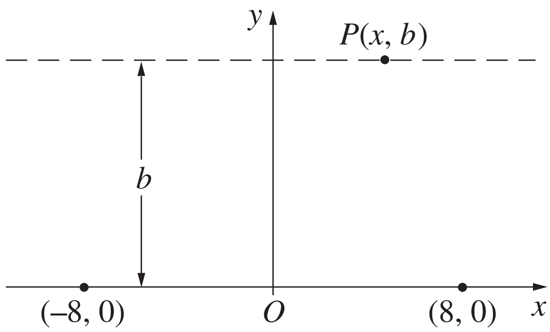
Show that $r = 20\sin\frac{\theta}{2}$ for $\frac{\pi}{2} < \theta < \pi$.

(iii) Sketch the graph of r, as defined by the equations in parts (i) and (ii), for $0 < \theta < \pi$. **3**

Question 10 continues on page 14

Question 10 (continued)

(b) On a dark night, two ships, Saga and Hero, sail parallel to a straight coastline on which there are two lights of equal brightness, 16 kilometres apart.



Suppose the coastline is represented by the x axis where the origin O is chosen to be the midpoint of the light sources. It is known that the (total) brightness from the lights on a ship at point $P(x, b)$ is

$$I = \frac{1}{b^2 + (x+8)^2} + \frac{1}{b^2 + (x-8)^2} \ .$$

(i) Show that $\dfrac{dI}{dx} = -\dfrac{2P}{Q}$ where **2**

$$P = \left[(x+8)\left(b^2 + (x-8)^2\right)^2 + (x-8)\left(b^2 + (x+8)^2\right)^2 \right]$$

and $Q = \left(b^2 + (x+8)^2\right)^2 \left(b^2 + (x-8)^2\right)^2$.

To answer parts (ii) and (iii), you may assume the following factorisation, given by a computer package, that

$$P = 2x\left(x^2 + 64 + b^2 + 16\sqrt{64+b^2}\right)\left(x^2 + 64 + b^2 - 16\sqrt{64+b^2}\right).$$

(ii) Saga sails parallel to the coast at a distance 15 km from the coast. **2**

By considering $\dfrac{dI}{dx}$, show that, as Saga sails from left to right, the brightness on Saga increases to a maximum when $x = 0$ and then decreases.

(iii) Hero sails parallel to the coast at a distance 6 km from the coast. **2**

Describe how the brightness on Hero changes as Hero sails from left to right. Give clear reasons for your answer.

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$