BOARD OF STUDIES
NEW SOUTH WALES

2003

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $e^{-3.5}$ correct to 3 significant figures. **2**

(b) Differentiate $3x + \tan x$ with respect to x. **2**

(c) An arc of length 5 units subtends an angle θ at the centre of a circle of radius 3 units. Find the value of θ to the nearest degree. **2**

(d) Iain and Louise paid \$315.00 for a meal at a restaurant. This included a $12\frac{1}{2}\%$ tip. What was the cost of the meal without the tip? **2**

(e) Find a primitive of $3x^2 - 8$. **2**

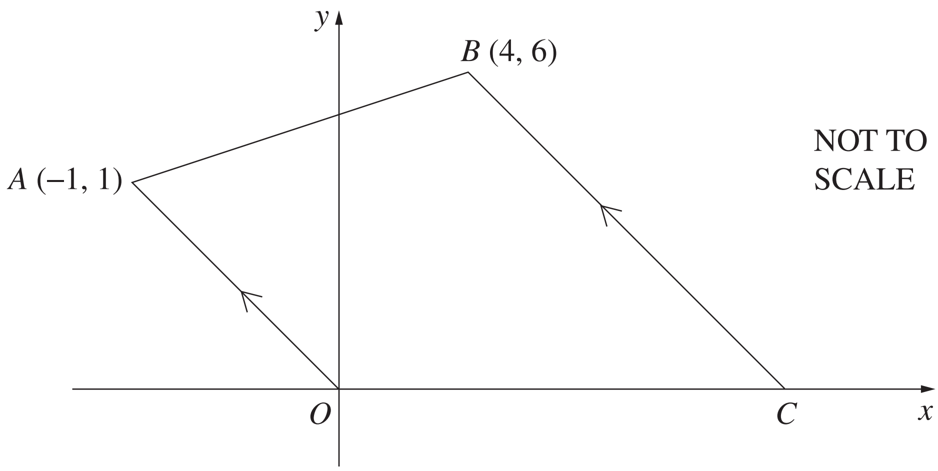
(f) Solve $|x - 3| = 7$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Find the equation of the normal to the curve $y = 2\log_e x$ at the point $(e, 2)$. **3**

(b)



In the diagram, $OABC$ is a trapezium with $OA \parallel CB$. The coordinates of O, A and B are $(0, 0)$, $(-1, 1)$ and $(4, 6)$ respectively.

(i) Calculate the length of OA. **1**

(ii) Write down the gradient of the line OA. **1**

(iii) What is the size of $\angle AOC$? **1**

(iv) Find the equation of the line BC, and hence find the coordinates of C. **2**

(v) Show that the perpendicular distance from O to the line BC is $5\sqrt{2}$. **2**

(vi) Hence, or otherwise, calculate the area of the trapezium $OABC$. **2**

Marks

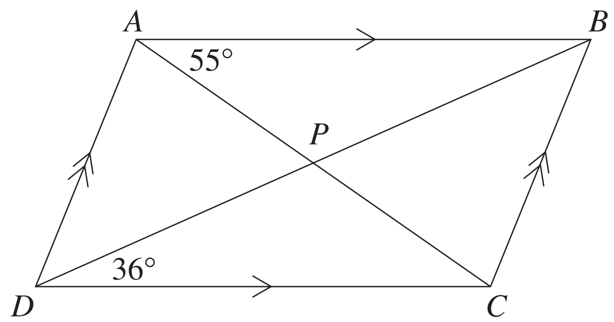
Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

(i) $\left(2e^{x}-4\right)^{9}$ **2**

(ii) $x^{2}\sin x$. **2**

(b) **2**



NOT TO SCALE

In the diagram, $ABCD$ is a parallelogram whose diagonals intersect at P. Given that $\angle BAC = 55°$ and $\angle CDB = 36°$, find the size of $\angle DPC$. Give reasons for your answer.

(c) Shade the region in the Cartesian plane for which the inequalities $y < x - 2$, $y \geq 0$ and $x \geq 6$ hold simultaneously. **3**

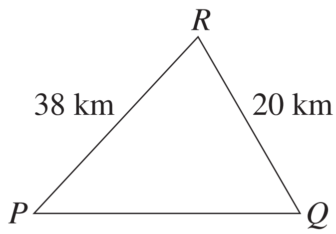
(d) (i) Find $\displaystyle\int \frac{2x}{x^{2}+5}\,dx$. **1**

(ii) Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^{2} x\,dx$. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a)



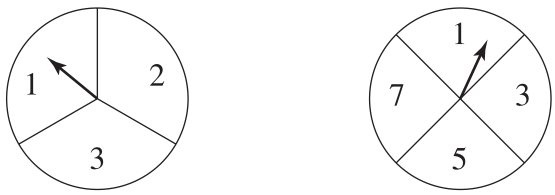
NOT TO SCALE

In the diagram, the point Q is due east of P. The point R is 38 km from P and 20 km from Q. The bearing of R from Q is 325°.

(i) What is the size of $\angle PQR$? **1**

(ii) What is the bearing of R from P? **3**

(b) The diagram shows two spinners which are spun simultaneously.

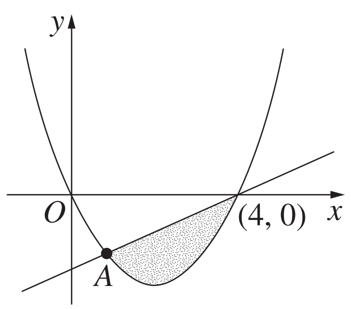


Each of the three outcomes on the first spinner are equally likely, and each of the four outcomes on the second spinner are equally likely.

(i) What is the probability that both spinners stop on the same number? **2**

(ii) What is the probability that at least one of the spinners stops on a 3? **1**

(c)



The graphs of $y = x - 4$ and $y = x^2 - 4x$ intersect at the points (4, 0) and A, as shown in the diagram.

(i) Find the coordinates of A. **2**

(ii) Find the area of the shaded region bounded by $y = x^2 - 4x$ and $y = x - 4$. **3**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = x^4 - 4x^3$.

(i) Show that $f'(x) = 4x^2(x-3)$. **1**

(ii) Find the coordinates of the stationary points of the curve $y = f(x)$, and determine their nature. **3**

(iii) Sketch the graph of the curve $y = f(x)$, showing the stationary points. **1**

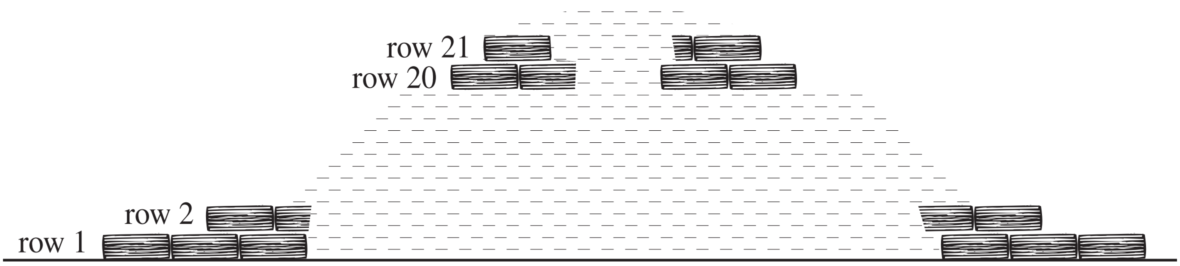
(iv) Find the values of x for which the graph of $y = f(x)$ is concave down. **2**

(b) A wall is built to stop erosion when a level road is cut through a hill.

The rows of the wall, built from concrete blocks 1.5 m long, are numbered from the bottom. The bottom row (row 1) is 180 m long.

Each of the rows 2, 3, …, 20 has 3 fewer blocks than the row below it.

Above row 20, each row has 1 block fewer than the row below it. The top row has 10 blocks.



(i) How many blocks are in row 20? **2**

(ii) What is the total number of rows in the wall? **1**

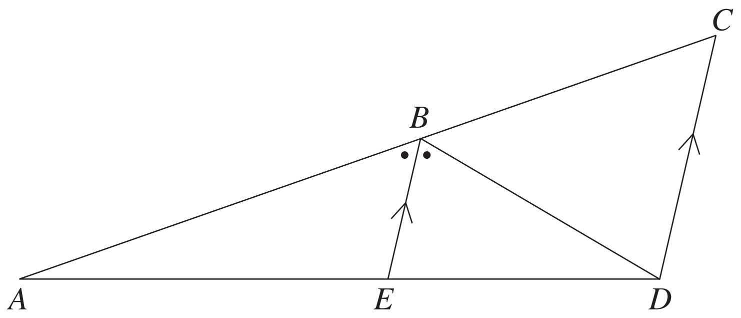
(iii) How many blocks are used in the construction of the wall? **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\log_2(3x-4)=5$. **2**

(b)



NOT TO SCALE

In the diagram, $BE \parallel CD$ and BE bisects $\angle ABD$.

Copy or trace the diagram into your writing booklet.

(i) Explain why $\angle EBD = \angle BDC$. **1**

(ii) Prove that ΔBCD is isosceles. **2**

(iii) Hence show that $AE : ED = AB : BD$. **2**

(c) A farmer accidentally spread a dangerous chemical on a paddock. The concentration of the chemical in the soil was initially measured at 5 kg/ha. One year later the concentration was found to be 2.8 kg/ha.

It is known that the concentration, C, is given by

$$C = C_0 e^{-kt},$$

where C_0 and k are constants, and t is measured in years.

(i) Evaluate C_0 and k. **3**

(ii) It is safe to use the paddock when the concentration is below 0.2 kg/ha. How long must the farmer wait after the accident before the paddock can be used? Give your answer in years, correct to one decimal place. **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Find the limiting sum of the geometric series **2**

$$2+\frac{2}{\sqrt{2}+1}+\frac{2}{\left(\sqrt{2}+1\right)^2}+\ldots$$

(ii) Explain why the geometric series **1**

$$2+\frac{2}{\sqrt{2}-1}+\frac{2}{\left(\sqrt{2}-1\right)^2}+\ldots$$

does NOT have a limiting sum.

(b) The velocity of a particle is given by

$$v=2-4\cos t \quad \text{for} \quad 0\le t\le 2\pi,$$

where v is measured in metres per second and t is measured in seconds.

(i) At what times during this period is the particle at rest? **2**

(ii) What is the maximum velocity of the particle during this period? **2**

(iii) Sketch the graph of v as a function of t for $0\le t\le 2\pi$. **2**

(iv) Calculate the total distance travelled by the particle between $t=0$ and $t=\pi$. **3**

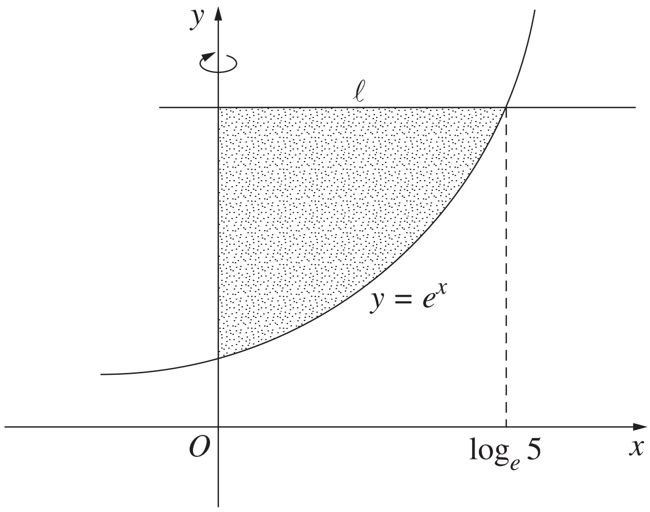
Marks

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) Write down the equation of the directrix of the parabola **1**

$$x^2 = -8y.$$

(b) **3**



In the diagram, the shaded region is bounded by the y-axis, the curve $y = e^x$ and a horizontal line ℓ that cuts the curve at a point whose x coordinate is $\log_e 5$.

A solid is formed by rotating the shaded region about the y-axis.

Write down a definite integral whose value is the volume of the solid. (Do NOT evaluate the integral.)

(c) Use Simpson's rule with three function values to find an approximation for **3**

$$\int_2^6 \frac{x}{\log_e x}\, dx\ .$$

Give your answer correct to one decimal place.

(d) (i) Show that for all values of m, the line $y = mx - 3m^2$ touches the parabola $x^2 = 12y$. **2**

(ii) Find the values of m for which this line passes through the point (5, 2). **2**

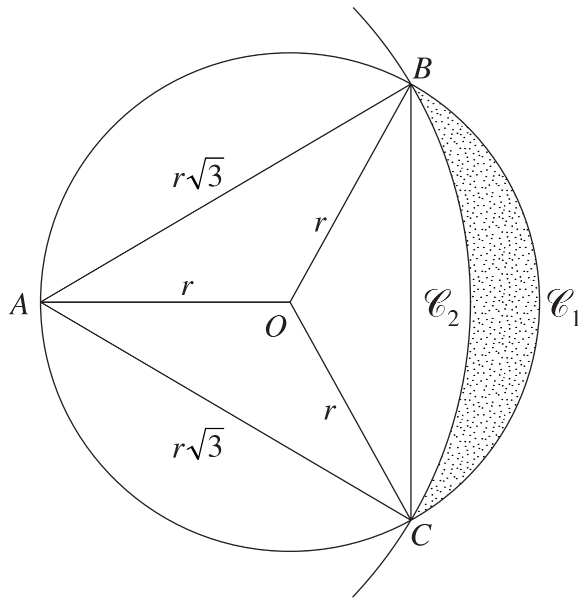
(iii) Hence determine the equations of the two tangents to the parabola $x^2 = 12y$ from the point (5, 2). **1**

Marks

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $2\sin^2 x - 3\sin x - 2 = 0$ for $0 \le x \le 2\pi$. **2**

(b) The point A lies on the circle $\mathscr{C}_1$ with centre O and radius r. The circle $\mathscr{C}_2$ with centre A and radius $r\sqrt{3}$ intersects $\mathscr{C}_1$ at the points B and C, as shown in the diagram.



(i) Using the cosine rule (or otherwise) find $\angle BAO$. **1**

(ii) Find the area of the sector BAC of $\mathscr{C}_2$. **1**

(iii) Find the area of the sector BOC of $\mathscr{C}_1$. **1**

(iv) Hence (or otherwise) find the area of the shaded region. **2**

Question 9 continues on page 11

Marks

Question 9 (continued)

(c) A fish is swimming at a constant speed against the current. The current is moving at a constant speed of u m s^{-1}. The speed of the fish relative to the water is v m s^{-1}, so that the actual speed of the fish is $(v-u)$ m s^{-1}.

The rate at which the fish uses energy is proportional to v^3, so the amount of energy used in t seconds is given by

$$E = av^3t,$$

where a is a constant.

(i) Show that the energy used to swim L metres is given by **1**

$$E = \frac{aLv^3}{v-u}.$$

(ii) Migrating fish try to minimise the total energy used to swim a fixed distance. Find the value of v that minimises E. (You may assume $v > u > 0$.) **4**

End of Question 9

Please turn over

Marks

Question 10 (12 marks) Use a SEPARATE writing booklet.

(a) Barbara borrows \$120 000 to be repaid over a period of 25 years at 6% per annum reducible interest. Each year there are k regular repayments of \$$F$. Interest is calculated and charged just before each repayment.

(i) Write down an expression for the amount owing after two repayments. **1**

(ii) Show that the amount owing after n repayments is **2**

$$A_n = 120\,000\,\alpha^n - \frac{kF\left(\alpha^n - 1\right)}{0.06},$$

where $\alpha = 1 + \dfrac{0.06}{k}$.

(iii) Calculate the amount of each repayment if the repayments are made quarterly (ie. $k = 4$). **2**

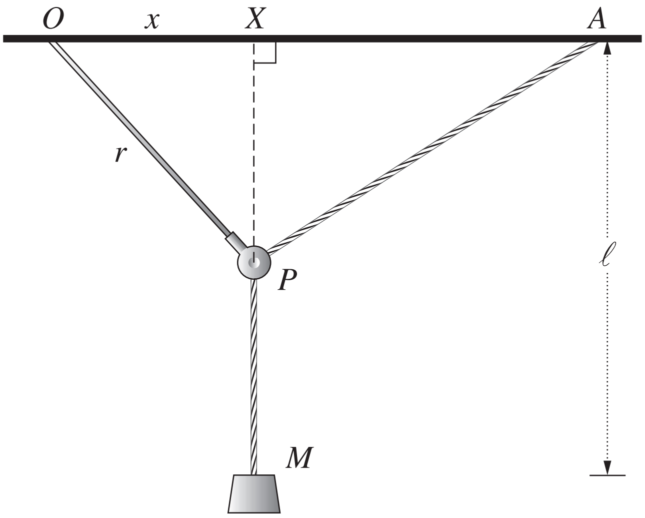
(iv) How much would Barbara have saved over the term of the loan if she had chosen to make monthly rather than quarterly repayments? **2**

Question 10 continues on page 13

Marks

Question 10 (continued)

(b) A pulley P is attached to the ceiling at O by a piece of metal that can swing freely. One end of a rope is attached to the ceiling at A. The rope is passed through the pulley P and a weight is attached to the other end of the rope at M, as shown in the diagram.



The distance OA is 1 m, the length of the rope is 2 m, and the length of the piece of metal $OP = r$ metres, where $0 < r < 1$. Let X be the point where the line MP produced meets OA. Let $OX = x$ metres and $XM = \ell$ metres.

(i) By considering triangles OXP and AXP, show that **1**

$$\ell = 2 + \sqrt{r^2 - x^2} - \sqrt{1 - 2x + r^2}\ .$$

(ii) Show that $\dfrac{d\ell}{dx} = \dfrac{\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right)}{\sqrt{r^2 - x^2}\sqrt{1 - 2x + r^2}\left(\sqrt{r^2 - x^2} + x\sqrt{1 - 2x + r^2}\right)}$. **2**

(iii) You are given the factorisation **2**

$$\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right) = (x - 1)\left(2x^2 - r^2x - r^2\right).$$

(Do NOT prove this.)

Find the value of x for which M is closest to the floor. Justify your answer.

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$