BOARD OF STUDIES

NEW SOUTH WALES

2005

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–8
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$. **2**

(b) (i) Find real numbers a and b such that $\displaystyle\frac{5x}{x^2-x-6} \equiv \frac{a}{x-3}+\frac{b}{x+2}$. **2**

(ii) Hence find $\displaystyle\int \frac{5x}{x^2-x-6}\,dx$. **1**

(c) Use integration by parts to evaluate $\displaystyle\int_1^e x^7 \log_e x\,dx$. **3**

(d) Using the table of standard integrals, or otherwise, find $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$. **2**

(e) Let $t=\tan\dfrac{\theta}{2}$.

(i) Show that $\dfrac{dt}{d\theta}=\dfrac{1}{2}\left(1+t^2\right)$. **1**

(ii) Show that $\sin\theta=\dfrac{2t}{1+t^2}$. **2**

(iii) Use the substitution $t=\tan\dfrac{\theta}{2}$ to find $\displaystyle\int \operatorname{cosec}\theta\,d\theta$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 3 + i$ and $w = 1 - i$. Find, in the form $x + iy$,

(i) $2z + iw$ **1**

(ii) $\bar{z}w$ **1**

(iii) $\dfrac{6}{w}$. **1**

(b) Let $\beta = 1 - i\sqrt{3}$.

(i) Express β in modulus-argument form. **2**

(ii) Express β^5 in modulus-argument form. **2**

(iii) Hence express β^5 in the form $x + iy$. **1**

(c) Sketch the region on the Argand diagram where the inequalities **3**

$$|z - \bar{z}| < 2 \quad \text{and} \quad |z - 1| \geq 1$$

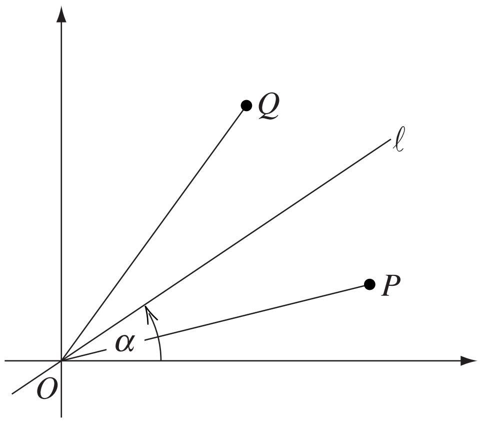
hold simultaneously.

Question 2 continues on page 5

Marks

Question 2 (continued)

(d) Let ℓ be the line in the complex plane that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.



The point P represents the complex number z_1, where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line ℓ to produce the point Q, which represents the complex number z_2. Hence $|z_1| = |z_2|$.

(i) Explain why $\arg(z_1) + \arg(z_2) = 2\alpha$. **2**

(ii) Deduce that $z_1 z_2 = |z_1|^2 (\cos 2\alpha + i \sin 2\alpha)$. **1**

(iii) Let $\alpha = \frac{\pi}{4}$ and let R be the point that represents the complex number $z_1 z_2$. **1**

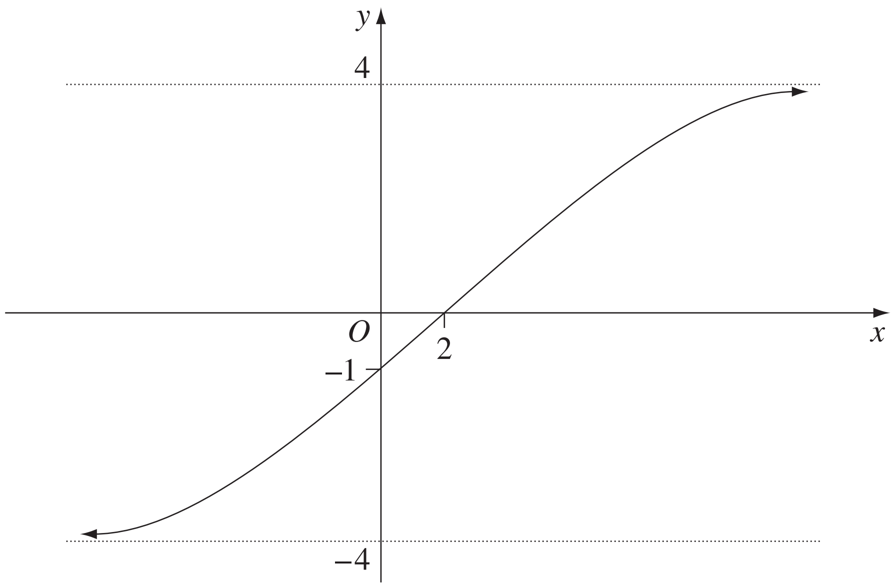
Describe the locus of R as z_1 varies.

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(x+3)$ **1**

(ii) $y = |f(x)|$ **1**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = f(|x|)$. **2**

(b) Sketch the graph of $y = x + \dfrac{8x}{x^2 - 9}$, clearly indicating any asymptotes and any points where the graph meets the axes. **4**

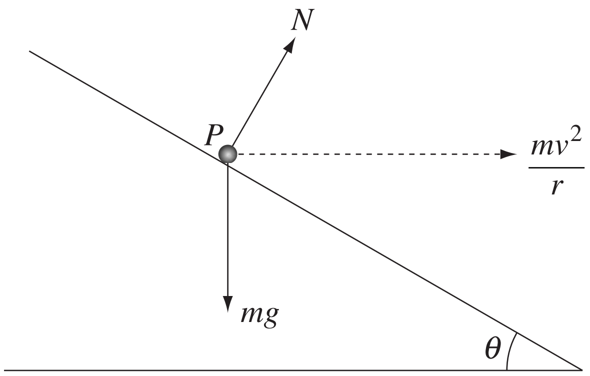
Question 3 continues on page 7

Marks

Question 3 (continued)

(c) Find the equation of the normal to the curve $x^3 - 4xy + y^3 = 1$ at $(2, 1)$. **3**

(d) **2**



The diagram shows the forces acting on a point P which is moving on a frictionless banked circular track. The point P has mass m and is moving in a horizontal circle of radius r with uniform speed v. The track is inclined at an angle θ to the horizontal. The point experiences a normal reaction force N from the track and a vertical force of magnitude mg due to gravity, so that the net force on the particle is a force of magnitude $\dfrac{mv^2}{r}$ directed towards the centre of the horizontal circle.

By resolving N in the horizontal and vertical directions, show that

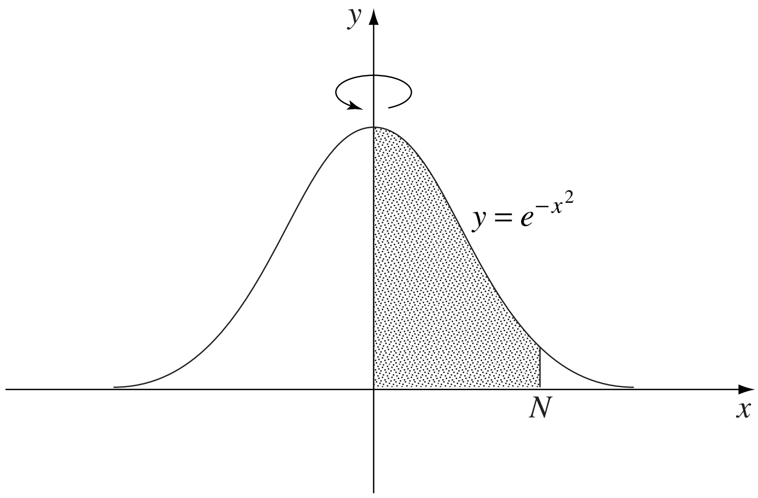
$$N = m\sqrt{g^2 + \frac{v^4}{r^2}}\ .$$

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a)



The shaded region between the curve $y = e^{-x^2}$, the x-axis, and the lines $x = 0$ and $x = N$, where $N > 0$, is rotated about the y-axis to form a solid of revolution.

(i) Use the method of cylindrical shells to find the volume of this solid in terms of N. **3**

(ii) What is the limiting value of this volume as $N \to \infty$? **1**

(b) Suppose α, β, γ and δ are the four roots of the polynomial equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

(i) Find the values of $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ in terms of p, q, r and s. **2**

(ii) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = p^2 - 2q$. **2**

(iii) Apply the result in part (ii) to show that $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ cannot have four real roots. **1**

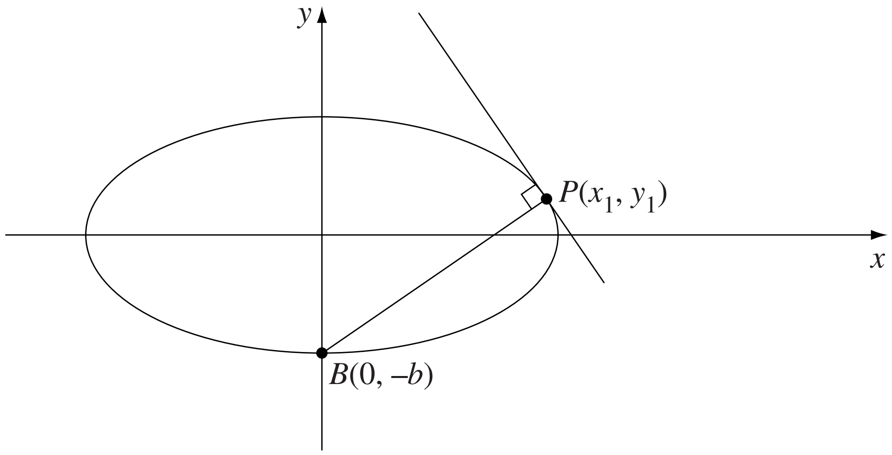
(iv) By evaluating the polynomial at $x = 0$ and $x = 1$, deduce that the polynomial equation $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ has exactly two real roots. **2**

Question 4 continues on page 9

Marks

Question 4 (continued)

(c)



The point $P(x_1, y_1)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$.

The equation of the normal to the ellipse at P is $a^2 y_1 x - b^2 x_1 y = (a^2 - b^2)\, x_1 y_1$.

(i) The normal at P passes through the point $B(0, -b)$. **2**

Show that $y_1 = \dfrac{b^3}{a^2 - b^2}$ or $y_1 = \pm b$.

(ii) Show that if $y_1 = \dfrac{b^3}{a^2 - b^2}$, the eccentricity of the ellipse is at least $\dfrac{1}{\sqrt{2}}$. **2**

End of Question 4

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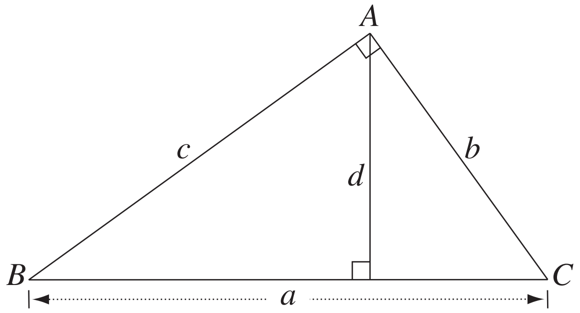
Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

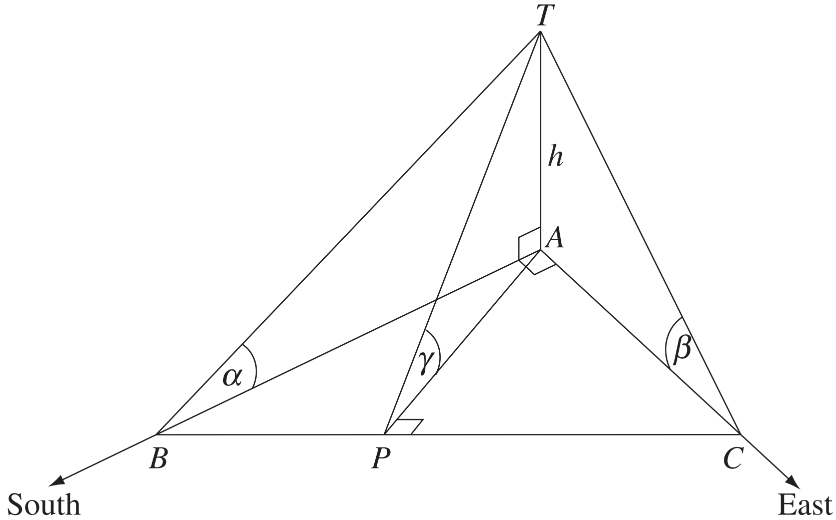
(a) (i) **1**



The triangle ABC is right-angled at A and has sides with lengths a, b and c, as shown in the diagram. The perpendicular distance from A to BC is d.

By considering areas, or otherwise, show that $b^2c^2 = d^2(b^2 + c^2)$.

(ii) **2**



The points A, B and C lie on a horizontal surface. The point B is due south of A. The point C is due east of A. There is a vertical tower, AT, of height h at A. The point P lies on BC and is chosen so that AP is perpendicular to BC.

Let α, β and γ denote the angles of elevation to the top of the tower from B, C and P respectively.

Using the result in part (i), or otherwise, show that $\tan^2\gamma = \tan^2\alpha + \tan^2\beta$.

Question 5 continues on page 12

Marks

Question 5 (continued)

(b) Mary and Ferdinand are competing against each other in a competition in which the winner is the first to score five goals. The outcome is recorded by listing, in order, the initial of the person who scores each goal. For example, one possible outcome would be recorded as *MFFMMMFMM*.

(i) Explain why there are five different ways in which the outcome could be recorded if Ferdinand scores only one goal in the competition. **1**

(ii) In how many different ways could the outcome of this competition be recorded? **2**

(c) Let $a > 0$ and let $f(x)$ be an increasing function such that $f(0) = 0$ and $f(a) = b$.

(i) Explain why $\int_0^a f(x)\,dx = ab - \int_0^b f^{-1}(x)\,dx$. **1**

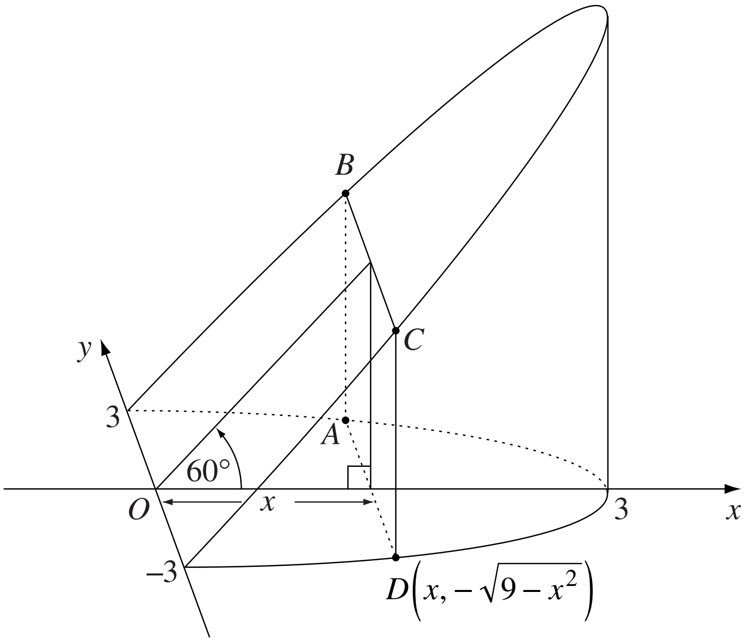
(ii) Hence, or otherwise, find the value of $\int_0^2 \sin^{-1}\left(\frac{x}{4}\right) dx$. **3**

Question 5 continues on page 13

Marks

Question 5 (continued)

(d) The base of a right cylinder is the circle in the xy-plane with centre O and radius 3. A wedge is obtained by cutting this cylinder with the plane through the y-axis inclined at 60° to the xy-plane, as shown in the diagram.



A rectangular slice $ABCD$ is taken perpendicular to the base of the wedge at a distance x from the y-axis.

(i) Show that the area of $ABCD$ is given by $2x\sqrt{27-3x^2}$. **2**

(ii) Find the volume of the wedge. **3**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) For each integer $n \geq 0$, let $I_n(x) = \int_0^x t^n e^{-t}\, dt$.

(i) Prove by induction that **4**

$$I_n(x) = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)\right].$$

(ii) Show that **1**

$$0 \leq \int_0^1 t^n e^{-t}\, dt \leq \frac{1}{n+1}.$$

(iii) Hence show that **1**

$$0 \leq 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \leq \frac{1}{(n+1)!}.$$

(iv) Hence find the limiting value of $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ as $n \to \infty$. **1**

Question 6 continues on page 15

Marks

Question 6 (continued)

(b) Let n be an integer greater than 2. Suppose ω is an nth root of unity and $\omega \neq 1$.

(i) By expanding the left-hand side, show that **2**

$$\left(1+2\omega+3\omega^2+4\omega^3+\cdots+n\omega^{n-1}\right)(\omega-1)=n.$$

(ii) Using the identity $\dfrac{1}{z^2-1} = \dfrac{z^{-1}}{z-z^{-1}}$, or otherwise, prove that **1**

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta},$$

provided that $\sin\theta \neq 0$.

(iii) Hence, if $\omega = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$, find the real part of $\dfrac{1}{\omega-1}$. **1**

(iv) Deduce that $1 + 2\cos\dfrac{2\pi}{5} + 3\cos\dfrac{4\pi}{5} + 4\cos\dfrac{6\pi}{5} + 5\cos\dfrac{8\pi}{5} = -\dfrac{5}{2}$. **1**

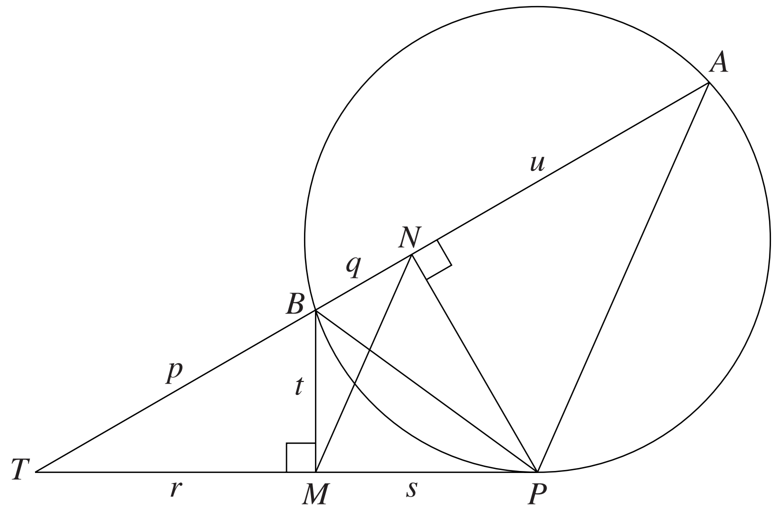
(v) By expressing the left-hand side of the equation in part (iv) in terms of $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{2\pi}{5}$, find the exact value, in surd form, of $\cos\dfrac{\pi}{5}$. **3**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a)



The points A, B and P lie on a circle.

The chord AB produced and the tangent at P intersect at the point T, as shown in the diagram. The point N is the foot of the perpendicular to AB through P, and the point M is the foot of the perpendicular to PT through B.

Copy or trace this diagram into your writing booklet.

(i) Explain why $BNPM$ is a cyclic quadrilateral. **1**

(ii) Prove that MN is parallel to PA. **3**

Let $TB = p$, $BN = q$, $TM = r$, $MP = s$, $MB = t$ and $NA = u$.

(iii) Show that $\frac{s}{u} < \frac{r}{p}$. **2**

(iv) Deduce that $s < u$. **1**

Question 7 continues on page 17

Marks

Question 7 (continued)

(b) The acceleration of any body towards the centre of a star due to the force of gravity is proportional to x^{-2}, where x is the distance of the body from the centre of the star. That is, $\ddot{x} = -\dfrac{k}{x^2}$, where k is a positive constant.

(i) A satellite is orbiting a star with constant speed, V, at a fixed distance R from the centre of the star. Its period of revolution is T. Use the fact that the satellite is moving in uniform circular motion to show that $k = \dfrac{4\pi^2 R^3}{T^2}$. **2**

(ii) The satellite is stopped suddenly. It then falls in a straight line towards the centre of the star under the influence of gravity. **3**

Show that the satellite's velocity, v, satisfies the equation

$$v^2 = \frac{8\pi^2 R^2}{T^2}\left(\frac{R-x}{x}\right),$$

where x is the distance of the satellite from the centre of the star.

(iii) Show that, if the mass of the star were concentrated at a single point, the satellite would reach this point after time $\dfrac{T}{4\sqrt{2}}$. **3**

You may assume that $\displaystyle\int \sqrt{\frac{x}{R-x}}\,dx = R\sin^{-1}\left(\sqrt{\frac{x}{R}}\right) - \sqrt{x(R-x)}$.

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose that a and b are positive real numbers, and let $f(x) = \dfrac{a+b+x}{3(abx)^{\frac{1}{3}}}$ for $x > 0$.

(i) Show that the minimum value of $f(x)$ occurs when $x = \dfrac{a+b}{2}$. **3**

(ii) Suppose that c is a positive real number. **2**

Show that $\left(\dfrac{a+b+c}{3\sqrt[3]{abc}}\right)^3 \geq \left(\dfrac{a+b}{2\sqrt{ab}}\right)^2$ and deduce that $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.

You may assume that $\dfrac{a+b}{2} \geq \sqrt{ab}$.

(iii) Suppose that the cubic equation $x^3 - px^2 + qx - r = 0$ has three positive real roots. Use part (ii) to prove that $p^3 \geq 27r$. **1**

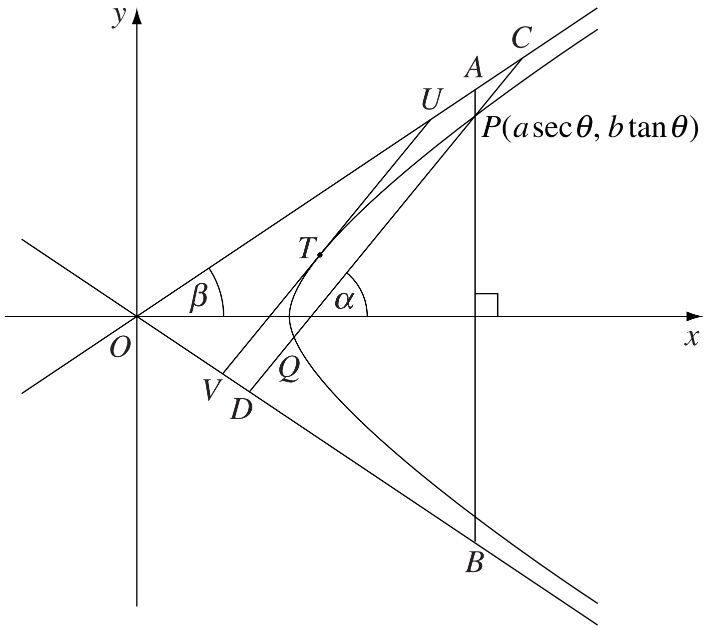
(iv) Deduce that the cubic equation $x^3 - 2x^2 + x - 1 = 0$ has exactly one real root. **2**

Question 8 continues on page 19

Marks

Question 8 (continued)

(b)



The point $P(a\sec\theta,\ b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The line through P perpendicular to the x-axis meets the asymptotes of the hyperbola, $\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$, at $A(a\sec\theta,\ b\sec\theta)$ and $B(a\sec\theta,\ -b\sec\theta)$ respectively. A second line through P, with gradient $\tan\alpha$, meets the hyperbola at Q and meets the asymptotes at C and D as shown. The asymptote $\frac{x}{a}-\frac{y}{b}=0$ makes an angle β with the x-axis at the origin, as shown.

(i) Show that $AP\times PB=b^2$. **1**

(ii) Show that $CP=\dfrac{AP\cos\beta}{\sin(\alpha-\beta)}$ and show that $PD=\dfrac{PB\cos\beta}{\sin(\alpha+\beta)}$. **2**

(iii) Hence deduce that $CP\times PD$ depends only on the value of α and not on the position of P. **1**

(iv) Let $CP=p$, $QD=q$ and $PQ=r$. **2**

Show that $p=q$.

(v) A tangent to the hyperbola is drawn at T parallel to CD. This tangent meets the asymptotes at U and V as shown. Show that T is the midpoint of UV. **1**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$