

B O A R D O F S T U D I E S
NEW SOUTH WALES

2006

**HIGHER SCHOOL CERTIFICATE
EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Total marks – 84

Attempt Questions 1–7

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{49 + x^2}$. **2**

(b) Using the substitution $u = x^4 + 8$, or otherwise, find $\int x^3 \sqrt{x^4 + 8} dx$. **3**

(c) Evaluate $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$. **2**

(d) Using the sum of two cubes, simplify: **2**

$$\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} - 1,$$

for $0 < \theta < \frac{\pi}{2}$.

(e) For what values of b is the line $y = 12x + b$ tangent to $y = x^3$? **3**

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = \sin^{-1}(x + 5)$.

(i) State the domain and range of the function $f(x)$. 2

(ii) Find the gradient of the graph of $y = f(x)$ at the point where $x = -5$. 2

(iii) Sketch the graph of $y = f(x)$. 2

(b) (i) By applying the binomial theorem to $(1+x)^n$ and differentiating, show that 1

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \dots + r\binom{n}{r}x^{r-1} + \dots + n\binom{n}{n}x^{n-1}.$$

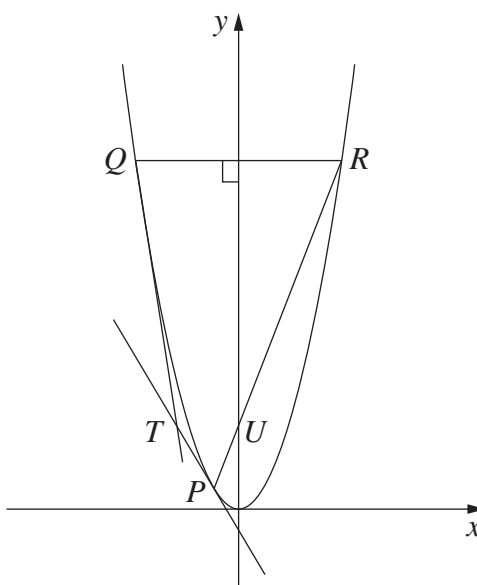
(ii) Hence deduce that 1

$$n3^{n-1} = \binom{n}{1} + \dots + r\binom{n}{r}2^{r-1} + \dots + n\binom{n}{n}2^{n-1}.$$

Question 2 continues on page 5

Question 2 (continued)

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ lie on the parabola $x^2 = 4ay$. The chord QR is perpendicular to the axis of the parabola. The chord PR meets the axis of the parabola at U .

The equation of the chord PR is $y = \frac{1}{2}(p+r)x - apr$. (Do NOT prove this.)

The equation of the tangent at P is $y = px - ap^2$. (Do NOT prove this.)

- (i) Find the coordinates of U . 1
- (ii) The tangents at P and Q meet at the point T . Show that the coordinates of T are $(a(p+q), apq)$. 2
- (iii) Show that TU is perpendicular to the axis of the parabola. 1

End of Question 2

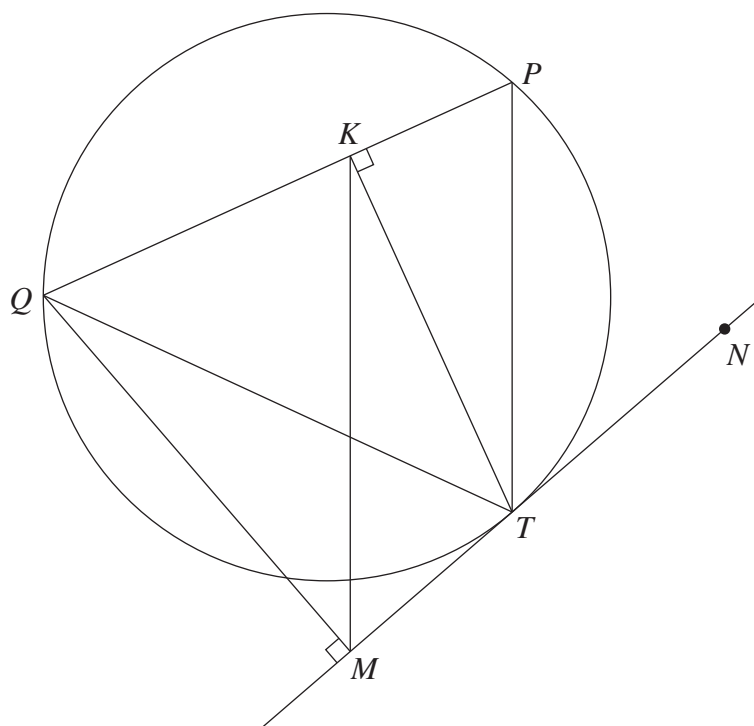
Question 3 (12 marks) Use a SEPARATE writing booklet.

- (a) Find $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$. **2**
- (b) (i) By considering $f(x) = 3 \log_e x - x$, show that the curve $y = 3 \log_e x$ and the line $y = x$ meet at a point P whose x -coordinate is between 1.5 and 2. **1**
- (ii) Use one application of Newton's method, starting at $x = 1.5$, to find an approximation to the x -coordinate of P . Give your answer correct to two decimal places. **2**
- (c) Sophie has five coloured blocks: one red, one blue, one green, one yellow and one white. She stacks two, three, four or five blocks on top of one another to form a vertical tower.
- (i) How many different towers are there that she could form that are three blocks high? **1**
- (ii) How many different towers can she form in total? **2**

Question 3 continues on page 7

Question 3 (continued)

(d)



The points P , Q and T lie on a circle. The line MN is tangent to the circle at T with M chosen so that QM is perpendicular to MN . The point K on PQ is chosen so that TK is perpendicular to PQ as shown in the diagram.

- (i) Show that $QKTM$ is a cyclic quadrilateral. 1
- (ii) Show that $\angle KMT = \angle KQT$. 1
- (iii) Hence, or otherwise, show that MK is parallel to TP . 2

End of Question 3

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Question 4 (12 marks) Use a SEPARATE writing booklet.

- (a) The cubic polynomial $P(x) = x^3 + rx^2 + sx + t$, where r , s and t are real numbers, has three real zeros, 1, α and $-\alpha$.
- (i) Find the value of r . 1
- (ii) Find the value of $s + t$. 2
- (b) A particle is undergoing simple harmonic motion on the x -axis about the origin. It is initially at its extreme positive position. The amplitude of the motion is 18 and the particle returns to its initial position every 5 seconds.
- (i) Write down an equation for the position of the particle at time t seconds. 1
- (ii) How long does the particle take to move from a rest position to the point halfway between that rest position and the equilibrium position? 2
- (c) A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$.
- Initially $x = -2$ and the velocity, v , is -6 .
- (i) Show that $v^2 = 9x^2(1+x)^2$. 2
- (ii) Hence, or otherwise, show that 2
- $$\int \frac{1}{x(1+x)} dx = -3t.$$
- (iii) It can be shown that for some constant c , 2
- $$\log_e \left(1 + \frac{1}{x} \right) = 3t + c. \quad (\text{Do NOT prove this.})$$

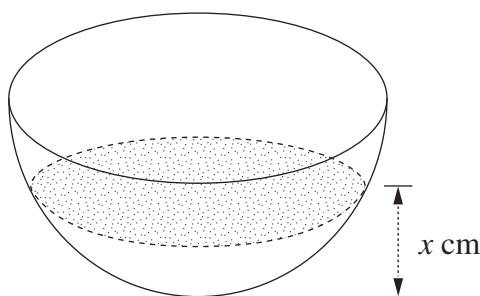
Using this equation and the initial conditions, find x as a function of t .

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Show that $y = 10e^{-0.7t} + 3$ is a solution of $\frac{dy}{dt} = -0.7(y - 3)$. 2

(b) Let $f(x) = \log_e(1 + e^x)$ for all x . Show that $f(x)$ has an inverse. 2

(c)



A hemispherical bowl of radius r cm is initially empty. Water is poured into it at a constant rate of k cm³ per minute. When the depth of water in the bowl is x cm, the volume, V cm³, of water in the bowl is given by

$$V = \frac{\pi}{3}x^2(3r - x). \quad (\text{Do NOT prove this.})$$

(i) Show that $\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}$. 2

(ii) Hence, or otherwise, show that it takes 3.5 times as long to fill the bowl 2

to the point where $x = \frac{2}{3}r$ as it does to fill the bowl to the point where

$$x = \frac{1}{3}r.$$

Question 5 continues on page 11

Question 5 (continued)

- (d) (i) Use the fact that $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$ to show that **1**

$$1 + \tan n\theta \tan(n+1)\theta = \cot \theta (\tan(n+1)\theta - \tan n\theta).$$

- (ii) Use mathematical induction to prove that, for all integers $n \geq 1$, **3**
- $$\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \cdots + \tan n\theta \tan(n+1)\theta = -(n+1) + \cot \theta \tan(n+1)\theta.$$

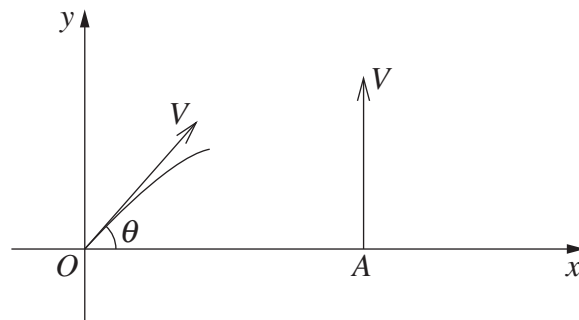
End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet.

- (a) Two particles are fired simultaneously from the ground at time $t=0$.

Particle 1 is projected from the origin at an angle θ , $0 < \theta < \frac{\pi}{2}$, with an initial velocity V .

Particle 2 is projected vertically upward from the point A , at a distance a to the right of the origin, also with an initial velocity of V .



It can be shown that while both particles are in flight, Particle 1 has equations of motion:

$$x = Vt \cos \theta$$
$$y = Vt \sin \theta - \frac{1}{2}gt^2 ,$$

and Particle 2 has equations of motion:

$$x = a$$
$$y = Vt - \frac{1}{2}gt^2 .$$

Do NOT prove these equations of motion.

Let L be the distance between the particles at time t .

Question 6 continues on page 13

Question 6 (continued)

- (i) Show that, while both particles are in flight, 2

$$L^2 = 2V^2t^2(1 - \sin\theta) - 2aVt\cos\theta + a^2.$$

- (ii) An observer notices that the distance between the particles in flight first decreases, then increases. 3

Show that the distance between the particles in flight is smallest when

$$t = \frac{a\cos\theta}{2V(1 - \sin\theta)} \text{ and that this smallest distance is } a\sqrt{\frac{1 - \sin\theta}{2}}.$$

- (iii) Show that the smallest distance between the two particles in flight occurs 1
while Particle 1 is ascending if $V > \sqrt{\frac{ag\cos\theta}{2\sin\theta(1 - \sin\theta)}}$.

- (b) In an endurance event, the probability that a competitor will complete the course is p and the probability that a competitor will not complete the course is $q = 1 - p$. Teams consist of either two or four competitors. A team scores points if at least half its members complete the course.

- (i) Show that the probability that a four-member team will have at least three of its members not complete the course is $4pq^3 + q^4$. 1
- (ii) Hence, or otherwise, find an expression in terms of q only for the probability that a four-member team will score points. 2
- (iii) Find an expression in terms of q only for the probability that a two-member team will score points. 1
- (iv) Hence, or otherwise, find the range of values of q for which a two-member team is more likely than a four-member team to score points. 2

End of Question 6

