BOARD OF STUDIES
NEW SOUTH WALES

2010
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{x}{\sqrt{1+3x^2}}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \tan x\,dx$. **3**

(c) Find $\int \frac{1}{x\left(x^2+1\right)}\,dx$. **3**

(d) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$. **4**

(e) Find $\int \frac{dx}{1+\sqrt{x}}$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 5 - i$.

(i) Find z^2 in the form $x + iy$. **1**

(ii) Find $z + 2\overline{z}$ in the form $x + iy$. **1**

(iii) Find $\dfrac{i}{z}$ in the form $x + iy$. **2**

(b) (i) Express $-\sqrt{3} - i$ in modulus–argument form. **2**

(ii) Show that $\left(-\sqrt{3} - i\right)^6$ is a real number. **2**

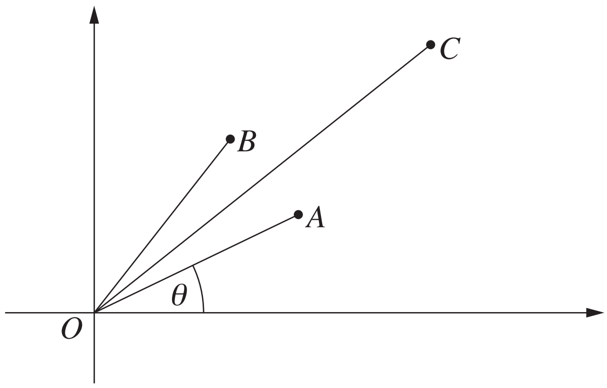
(c) Sketch the region in the complex plane where the inequalities $1 \le |z| \le 2$ and $0 \le z + \overline{z} \le 3$ hold simultaneously. **2**

Question 2 continues on page 5

Question 2 (continued)

(d) Let $z = \cos\theta + i\sin\theta$ where $0 < \theta < \frac{\pi}{2}$.

On the Argand diagram the point A represents z, the point B represents z^2 and the point C represents $z + z^2$.



Copy or trace the diagram into your writing booklet.

(i) Explain why the parallelogram $OACB$ is a rhombus. **1**

(ii) Show that $\arg\left(z + z^2\right) = \frac{3\theta}{2}$. **1**

(iii) Show that $\left|z + z^2\right| = 2\cos\frac{\theta}{2}$. **2**

(iv) By considering the real part of $z + z^2$, or otherwise, deduce that $\cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$. **1**

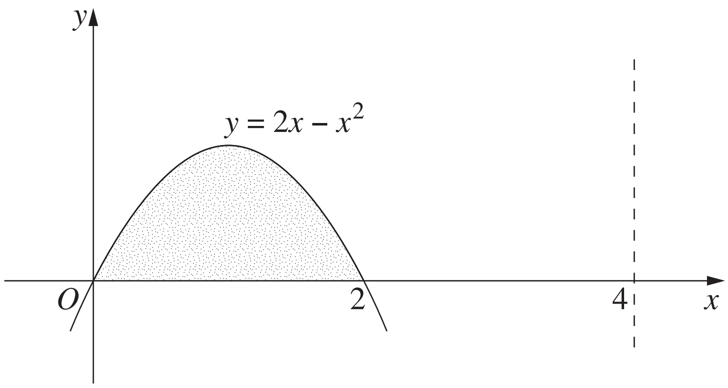
End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Sketch the graph $y = x^2 + 4x$. **1**

(ii) Sketch the graph $y = \dfrac{1}{x^2 + 4x}$. **2**

(b) The region shaded in the diagram is bounded by the x-axis and the curve $y = 2x - x^2$. **4**



The shaded region is rotated about the line $x = 4$.

Find the volume generated.

(c) Two identical biased coins are each more likely to land showing heads than showing tails. **2**

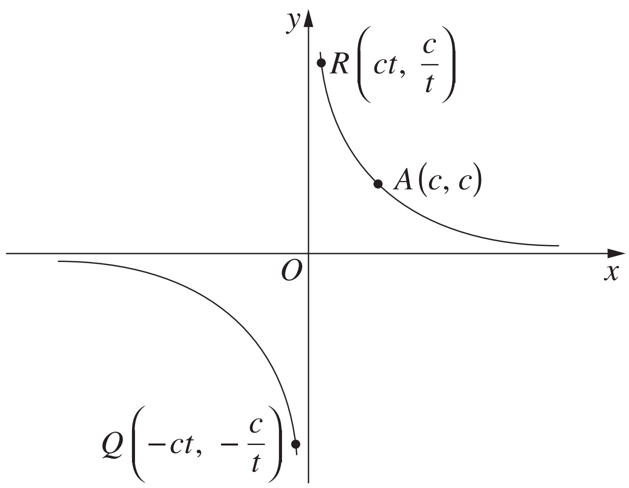
The two coins are tossed together, and the outcome is recorded. After a large number of trials it is observed that the probability that the two coins land showing a head and a tail is 0.48.

What is the probability that both coins land showing heads?

Question 3 continues on page 7

Question 3 (continued)

(d) The diagram shows the rectangular hyperbola $xy = c^2$, with $c > 0$.



The points $A(c, c)$, $R\left(ct, \frac{c}{t}\right)$ and $Q\left(-ct, -\frac{c}{t}\right)$ are points on the hyperbola, with $t \neq \pm 1$.

(i) The line ℓ_1 is the line through R perpendicular to QA. **2**

Show that the equation of ℓ_1 is

$$y = -tx + c\left(t^2 + \frac{1}{t}\right).$$

(ii) The line ℓ_2 is the line through Q perpendicular to RA. **1**

Write down the equation of ℓ_2.

(iii) Let P be the point of intersection of the lines ℓ_1 and ℓ_2. **2**

Show that P is the point $\left(\frac{c}{t^2}, ct^2\right)$.

(iv) Give a geometric description of the locus of P. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) (i) A curve is defined implicitly by $\sqrt{x} + \sqrt{y} = 1$. **2**

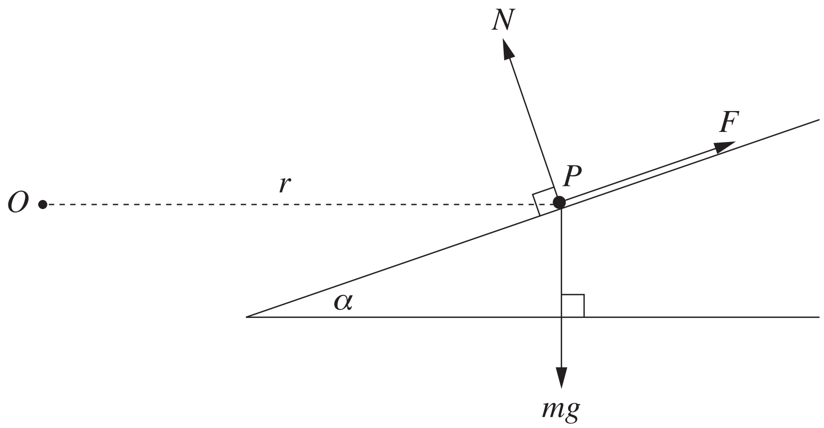
Use implicit differentiation to find $\dfrac{dy}{dx}$.

(ii) Sketch the curve $\sqrt{x} + \sqrt{y} = 1$. **2**

(iii) Sketch the curve $\sqrt{|x|} + \sqrt{|y|} = 1$. **1**

(b) A bend in a highway is part of a circle of radius r, centre O. Around the bend the highway is banked at an angle α to the horizontal.

A car is travelling around the bend at a constant speed v. Assume that the car is represented by a point P of mass m. The forces acting on the car are a lateral force F, the gravitational force mg and a normal reaction N to the road, as shown in the diagram.



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(i) By resolving forces, show that $F = mg\sin\alpha - \dfrac{mv^2}{r}\cos\alpha$. **3**

(ii) Find an expression for v such that the lateral force F is zero. **1**

Question 4 continues on page 9

Question 4 (continued)

(c) Let k be a real number, $k \geq 4$.

Show that, for every positive real number b, there is a positive real number a such that $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{k}{a+b}$. **3**

(d) A group of 12 people is to be divided into discussion groups.

(i) In how many ways can the discussion groups be formed if there are 8 people in one group, and 4 people in another? **1**

(ii) In how many ways can the discussion groups be formed if there are 3 groups containing 4 people each? **2**

End of Question 4

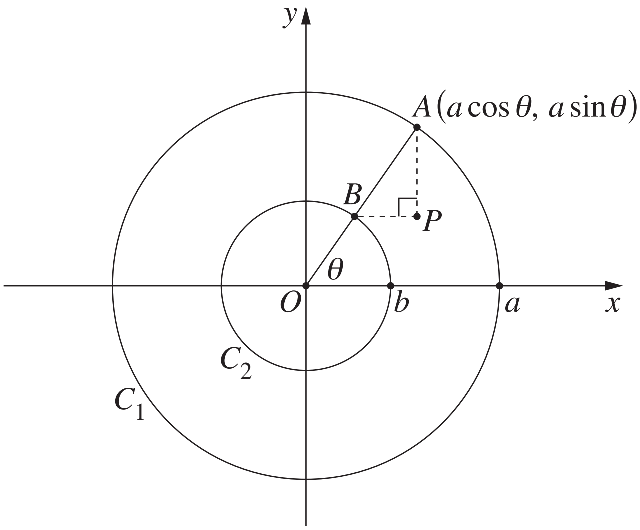
Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows two circles, C_1 and C_2, centred at the origin with radii a and b, where $a > b$.

The point A lies on C_1 and has coordinates $(a\cos\theta, a\sin\theta)$.

The point B is the intersection of OA and C_2.

The point P is the intersection of the horizontal line through B and the vertical line through A.



(i) Write down the coordinates of B. **1**

(ii) Show that P lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **1**

(iii) Find the equation of the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at P. **2**

(iv) Assume that A is not on the y-axis. **2**

Show that the tangent to the circle C_1 at A, and the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at P, intersect at a point on the x-axis.

Question 5 continues on page 11

Question 5 (continued)

(b) Show that **2**

$$\int \frac{dy}{y(1-y)} = \ln\left(\frac{y}{1-y}\right) + c$$

for some constant c, where $0 < y < 1$.

(c) A TV channel has estimated that if it spends \$$x$ on advertising a particular program it will attract a proportion $y(x)$ of the potential audience for the program, where

$$\frac{dy}{dx} = ay(1-y)$$

and $a > 0$ is a given constant.

(i) Explain why $\frac{dy}{dx}$ has its maximum value when $y = \frac{1}{2}$. **1**

(ii) Using part (b), or otherwise, deduce that **3**

$$y(x) = \frac{1}{ke^{-ax} + 1}$$

for some constant $k > 0$.

(iii) The TV channel knows that if it spends no money on advertising the program then the audience will be one-tenth of the potential audience. **1**

Find the value of the constant k referred to in part (c) (ii).

(iv) What feature of the graph $y = \frac{1}{ke^{-ax} + 1}$ is determined by the result in part (c) (i)? **1**

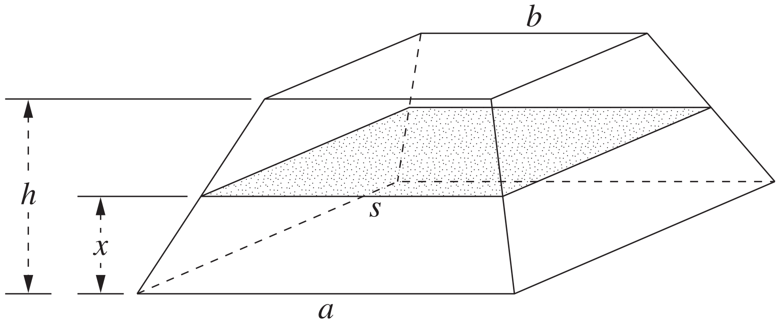
(v) Sketch the graph $y = \frac{1}{ke^{-ax} + 1}$. **1**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the frustum of a right square pyramid. (A frustum of a pyramid is a pyramid with its top cut off.)

The height of the frustum is h m. Its base is a square of side a m, and its top is a square of side b m (with $a > b > 0$).



A horizontal cross-section of the frustum, taken at height x m, is a square of side s m, shown shaded in the diagram.

(i) Show that $s = a - \frac{(a-b)}{h}x$. **2**

(ii) Find the volume of the frustum. **2**

(b) A sequence a_n is defined by **3**

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$, with $a_0 = a_1 = 2$.

Use mathematical induction to prove that

$$a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n \quad \text{for all } n \geq 0.$$

Question 6 continues on page 13

Question 6 (continued)

(c) (i) Expand $\left(\cos\theta + i\sin\theta\right)^5$ using the binomial theorem. **1**

(ii) Expand $\left(\cos\theta + i\sin\theta\right)^5$ using de Moivre's theorem, and hence show that **3**

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(iii) Deduce that $x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to **1**

$$16x^5 - 20x^3 + 5x - 1 = 0.$$

(iv) Find the polynomial $p(x)$ such that $(x-1)\,p(x) = 16x^5 - 20x^3 + 5x - 1$. **1**

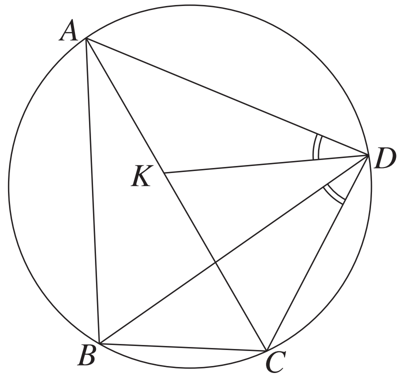
(v) Find the value of a such that $p(x) = (4x^2 + ax - 1)^2$. **1**

(vi) Hence find an exact value for $\sin\left(\frac{\pi}{10}\right)$. **1**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram $ABCD$ is a cyclic quadrilateral. The point K is on AC such that $\angle ADK = \angle CDB$, and hence ΔADK is similar to ΔBDC.

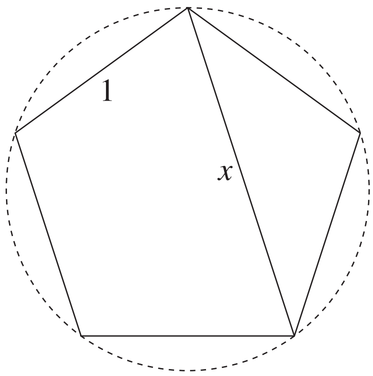


Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle ADB$ is similar to $\triangle KDC$. **2**

(ii) Using the fact that $AC = AK + KC$, show that $BD \times AC = AD \times BC + AB \times DC$. **2**

(iii) A regular pentagon of side length 1 is inscribed in a circle, as shown in the diagram. **2**



Let x be the length of a chord in the pentagon.

Use the result in part (ii) to show that $x = \dfrac{1+\sqrt{5}}{2}$.

Question 7 continues on page 15

Question 7 (continued)

(b) The graphs of $y = 3x - 1$ and $y = 2^x$ intersect at $(1, 2)$ and at $(3, 8)$. **1**

Using these graphs, or otherwise, show that $2^x \geq 3x - 1$ for $x \geq 3$.

(c) Let $P(x) = (n-1)x^n - nx^{n-1} + 1$, where n is an odd integer, $n \geq 3$.

(i) Show that $P(x)$ has exactly two stationary points. **1**

(ii) Show that $P(x)$ has a double zero at $x = 1$. **1**

(iii) Use the graph $y = P(x)$ to explain why $P(x)$ has exactly one real zero other than 1. **2**

(iv) Let α be the real zero of $P(x)$ other than 1. **2**

Using part (b), or otherwise, show that $-1 < \alpha \leq -\frac{1}{2}$.

(v) Deduce that each of the zeros of $4x^5 - 5x^4 + 1$ has modulus less than or equal to 1. **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

Let

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \quad \text{and} \quad B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx,$$

where n is an integer, $n \geq 0$. (Note that $A_n > 0$, $B_n > 0$.)

(a) Show that $n A_n = \dfrac{2n-1}{2} A_{n-1}$ for $n \geq 1$. **2**

(b) Using integration by parts on A_n, or otherwise, show that **1**

$$A_n = 2n \int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx \quad \text{for } n \geq 1.$$

(c) Use integration by parts on the integral in part (b) to show that **3**

$$\frac{A_n}{n^2} = \frac{(2n-1)}{n} B_{n-1} - 2B_n \quad \text{for } n \geq 1.$$

(d) Use parts (a) and (c) to show that **1**

$$\frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) \quad \text{for } n \geq 1.$$

(e) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$. **2**

(f) Use the fact that $\sin x \geq \dfrac{2}{\pi} x$ for $0 \leq x \leq \dfrac{\pi}{2}$ to show that **1**

$$B_n \leq \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx.$$

Question 8 continues on page 17

Question 8 (continued)

(g) Show that $\displaystyle\int_0^{\frac{\pi}{2}} x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx = \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx.$ **1**

(h) From parts (f) and (g) it follows that **2**

$$B_n \le \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx.$$

Use the substitution $x = \frac{\pi}{2}\sin t$ in this inequality to show that

$$B_n \le \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}} \cos^{2n+3} t\, dt \le \frac{\pi^3}{16(n+1)}A_n.$$

(i) Use part (e) to deduce that **1**

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \le \sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}.$$

(j) What is $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}$? **1**

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$