BOARD OF STUDIES
NEW SOUTH WALES

2010 HSC Mathematics Extension 2 Sample Answers

This document contains 'sample answers', or, in the case of some questions, 'answers could include'. These are developed by the examination committee for two purposes. The committee does this:

(a) as part of the development of the examination paper to ensure the questions will effectively assess students' knowledge and skills, and

(b) in order to provide some advice to the Supervisor of Marking about the nature and scope of the responses expected of students.

The 'sample answers' or similar advice are not intended to be exemplary or even complete answers or responses. As they are part of the examination committee's 'working document', they may contain typographical errors, omissions, or only some of the possible correct answers.

Question 1 (a)

Use the substitution $u = 1 + 3x^2$

$du = 6x\,dx$, so

$$\int \frac{x}{\sqrt{1+3x^2}}\,dx = \frac{1}{6}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{6}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{2}{6}u^{\frac{1}{2}} = \frac{1}{3}\sqrt{1+3x^2} + C$$

Question 1 (b)

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$$

$$= -\ln(\cos x)\ \Big|_0^{\frac{\pi}{4}}$$

$$= -\ln\frac{1}{\sqrt{2}} + \ln 1 = \ln\sqrt{2}$$

Question 1 (c)

Use partial fraction:

$$\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{b+cx}{1+x^2}$$

$$= \frac{a(1+x^2) + bx + cx^2}{x(1+x^2)}$$

Want $1 = a(1+x^2) + bx + cx^2$ for $x \in R$

$x = 0: \ 1 = a$

$$\left.\begin{array}{r} x = 1: \ 1 = 2 + b + c \\ x = -1: \ 1 = 2 - b + c \end{array}\right\} \begin{array}{l} -2 = 2c, \text{ so } \quad c = -1 \\ 0 = 2b, \text{ so } \quad b = 0 \end{array}$$

$$\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(1+x^2) + C$$

Question 1 (d)

Note $\sin x = \dfrac{2t}{1+t^2}, dx = \dfrac{2dt}{1+t^2}$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = \int_0^{\tan\frac{\pi}{4}} \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{1+t^2+2t}\,dt$$

$$= \int_0^1 \frac{2}{(1+t)^2}\,dt = -\frac{2}{1+t}\Bigg|_0^1$$

$$= -1 + 2 = 1$$

Question 1 (e)

Use the substitution $u = 1 + \sqrt{x}$

Then

$$du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2(u-1)}dx$$

$$\int \frac{1}{1+\sqrt{x}}dx = \int \frac{2(u-1)}{u}du$$

$$= 2\int\left(1 - \frac{1}{u}\right)du = 2u - 2\ln u + C$$

$$= 2\left(1 + \sqrt{x} - \ln\left(1+\sqrt{x}\right)\right) + C$$

Question 2 (a) (i)

$$\begin{aligned} z^2 &= (5-i)^2 \\ &= 25 - 10i - 1 \\ &= 24 - 10i \end{aligned}$$

Question 2 (a) (ii)

$$\begin{aligned} z + 2\overline{z} &= 5 - i + 2(5+i) \\ &= 15 + i \end{aligned}$$

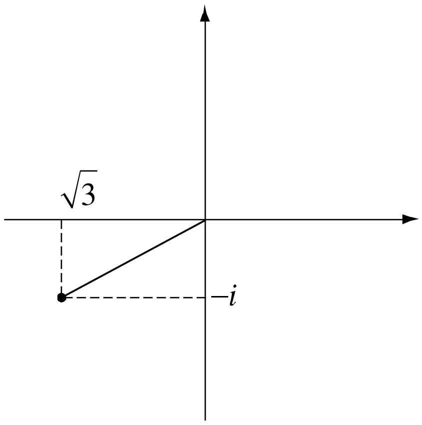
Question 2 (a) (iii)

$$\frac{i}{z} = \frac{i}{5-i} = \frac{(5+i)i}{25+1}$$

$$= \frac{-1+5i}{26} = -\frac{1}{26} + \frac{5}{26}i$$

Question 2 (b) (i)

$\left|-\sqrt{3}-i\right| = \sqrt{3+1} = 2$



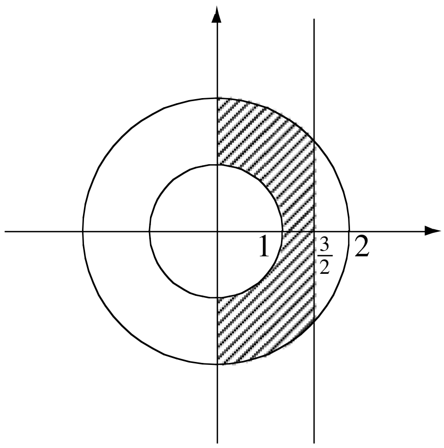
$\arg\left(-\sqrt{3}-i\right) = -\dfrac{5\pi}{6}$

$-\sqrt{3}-i = 2\operatorname{cis}\left(-\dfrac{5\pi}{6}\right)$

Question 2 (b) (ii)

$\left(-\sqrt{3}-i\right)^6 = 2^6\operatorname{cis}\left(-5\pi\right) = -2^6$ which is real.

Question 2 (c)



$0 \le z + \overline{z} = 2\operatorname{Re} z \le 3$

so $\quad 0 \le \operatorname{Re} z \le \dfrac{3}{2}$

Question 2 (d) (i)

$|z| = \cos^2\theta + \sin^2\theta = 1$, so

$\left|z^2\right| = |z|^2 = 1$

Hence $OA = OB$

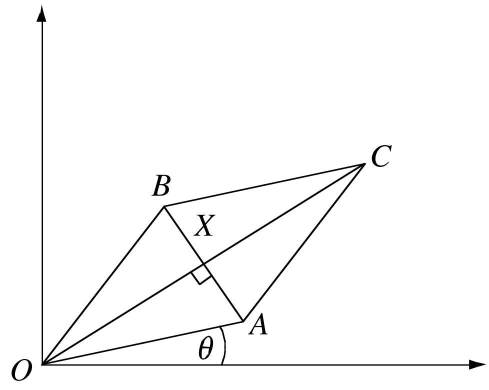
Question 2 (d) (ii)

$\arg z = \theta, \quad \arg z^2 = 2\theta$

since OC bisects $\angle AOB$ from (i)

$\arg\left(z + z^2\right) = \dfrac{\theta + 2\theta}{2} = \dfrac{3\theta}{2}$

Question 2 (d) (iii)



Let X be the midpoint of OC.

$\dfrac{OX}{OA} = \cos\dfrac{\theta}{2}$, so

$OC = 2\cos\dfrac{\theta}{2}$,

Hence $\left|z + z^2\right| = OC = 2\cos\dfrac{\theta}{2}$

(Note: by (i) $\angle OXA$ is 90°)

Question 2 (d) (iv)

Since $\arg z = \theta$ and $\arg z^2 = 2\theta$ we get

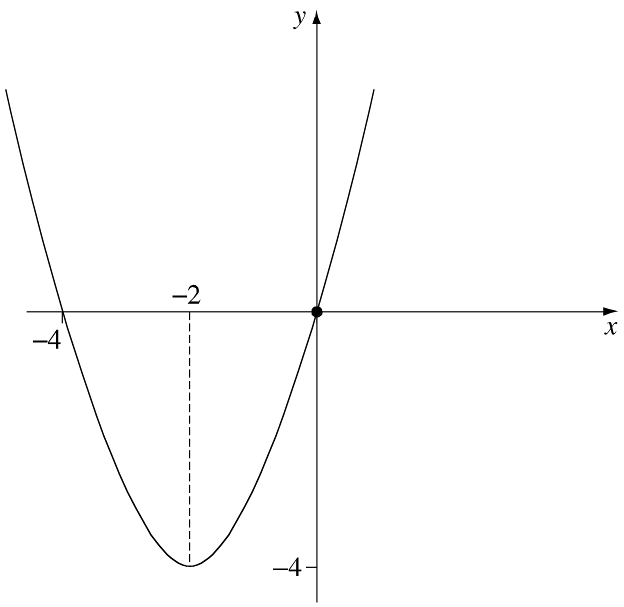
$\text{Re}\left(z + z^2\right) = \cos\theta + \cos 2\theta$

On the other hand, from parts (ii) and (iii)

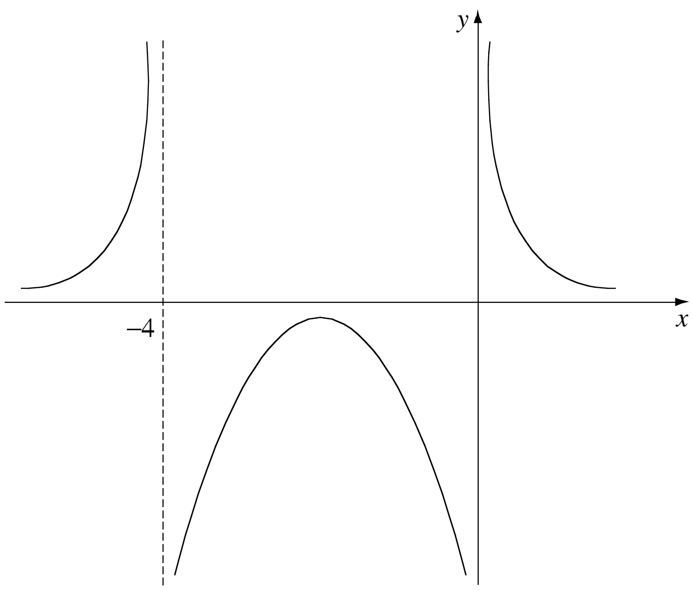
$$\text{Re}\left(z + z^2\right) = \left|z + z^2\right| \cos\left(\arg\left(z + z^2\right)\right)$$

$$= 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Question 3 (a) (i)



Question 3 (a) (ii)



Question 3 (b)

Use cylindrical shells:

$$\text{Volume} = 2\pi \int_0^2 y(4-x)\,dx$$

$$= 2\pi \int_0^2 (2x - x^2)(4-x)\,dx$$

$$= 2\pi \int_0^2 x^3 - 6x^2 + 8x\,dx$$

$$= 2\pi \left(\frac{x^4}{4} - \frac{6}{3}x^3 + \frac{8}{2}x^2 \right)\Bigg|_0^2$$

$$= 2\pi \left(\frac{x^4}{4} - 2x^3 + 4x^2 \right)\Bigg|_0^2$$

$$= 2\pi(4 - 16 + 16)$$

$$= 8\pi$$

Question 3 (c)

Let
p = probability for coin to show head
$1 - p$ = probability for coin to show tail
It is given that probability for one showing head and one showing tail is 0.48, that is,
$2p(1-p) = 0.48$.
Solving quadratic for p gives $p = 0.6$
Since it is given that it is more likely that the coin shows head (therefore excluding the other solution 0.4)
The probability showing two heads is $p^2 = 0.36$.

Question 3 (d) (i)

Slope of line QA:

$$M_{QA} = \frac{c + \frac{c}{t}}{c + ct} = \frac{t+1}{t(1+t)} = \frac{1}{t}$$

Hence the slope of ℓ_1 is $-t$, so its equation is

$$y = -tx + k$$

for some k. Since $R\left(ct, \frac{c}{t}\right)$ is on ℓ_1

$$\frac{c}{t} = -t^2c + k, \text{ so } k = c\left(\frac{1}{t} + t^2\right)$$

and so the equation for ℓ_1 is

$$y = -tx + c\left(\frac{1}{t} + t^2\right)$$

Question 3 (d) (ii)

Substitute $-t$ for t in the equation from part (i):

$$y = tx + c\left(t^2 - \frac{1}{t}\right)$$

Question 3 (d) (iii)

Solve the simultaneous equations

$$y = -tx + c\left(t^2 + \frac{1}{t}\right)$$

$$y = tx + c\left(t^2 - \frac{1}{t}\right)$$

$$2tx = c\left(t^2 + \frac{1}{t}\right) - c\left(t^2 - \frac{1}{t}\right)$$

$$= 2c\frac{1}{t}$$

Hence $x = \frac{c}{t^2}$ and so

$$y = -t\frac{c}{t^2} + c\left(t^2 + \frac{1}{t}\right) = ct^2$$

The point of intersection therefore is

$$\left(\frac{c}{t^2},\ ct^2\right)$$

Question 3 (d) (iv)

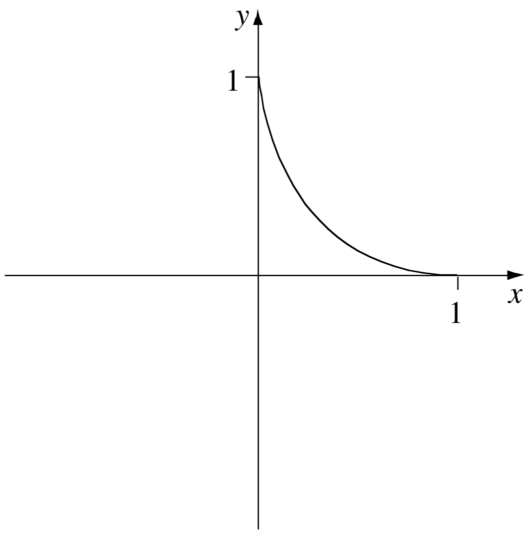
It describes the branch of the hyperbola $xy = c^2$ in the first quadrant (excluding A).

Question 4 (a) (i)

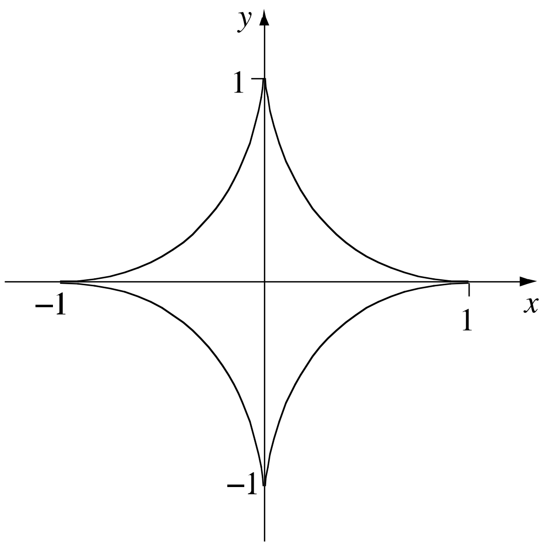
$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

Question 4 (a) (ii)



Question 4 (a) (iii)



Question 4 (b) (i)

Resolving vertically we get $mg = F\sin\alpha + N\cos\alpha$
Resolving horizontally we get

$F\cos\alpha = N\sin\alpha - \frac{mv^2}{r}$

So,

$mg\sin\alpha = F\sin^2\alpha + N\cos\alpha\sin\alpha$

$F\cos^2\alpha = N\cos\alpha\sin\alpha - \frac{mv^2}{r}\cos\alpha$

Subtracting

$mg\sin\alpha - F\cos^2\alpha = F\sin^2\alpha + \frac{mv^2}{r}\cos\alpha$

So $F = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$

Question 4 (b) (ii)

$F = 0$ means

$mg\sin\alpha - \frac{mv^2}{r}\cos\alpha = 0$

That is,

$\frac{v^2}{r}\cos\alpha = g\sin\alpha$

So $v^2 = rg\tan\alpha$

$\therefore v = \sqrt{rg\tan\alpha}$

Question 4 (c)

Multiply the identity by $ab(a + b)$, so

$$(a+b)^2 = kab$$

$$a^2 + 2ab + b^2 = kab$$

$$a^2 + (2-k)ab + b^2 = 0$$

If we fix $b > 0$, then the above is a quadratic in a. To be able to solve it we need to check that the discriminant Δ is non-negative:

$$\Delta = (2-k)^2 b^2 - 4b^2 = (4 - 4k + k^2)b^2 - 4b^2$$
$$= b^2 k(k-4) \geq 0$$

since it is given that $k \geq 4$

Now

$$a = \frac{1}{2}\left(b(k-2) + \sqrt{\Delta}\right)$$

is a solution. Since $b > 0$, $k - 2 > 0$ and $\sqrt{\Delta} \geq 0$, there is a positive solution a for every $b > 0$.

Question 4 (d) (i)

$$\binom{12}{8} \qquad \left(\text{or } \binom{12}{4}\right)$$

Question 4 (d) (ii)

$$\frac{1}{3!}\binom{12}{4} \times \binom{8}{4} \times \binom{4}{4}$$
$$= \frac{1}{6}\binom{12}{4} \times \binom{8}{4}$$

Question 5 (a) (i)

$$B = \left(b\cos\theta, b\sin\theta\right)$$

Question 5 (a) (ii)

P has coordinates $(a\cos\theta, b\sin\theta)$

Substitute into $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$\begin{aligned}\text{LHS} &= \frac{(a\cos\theta)^2}{a^2}+\frac{(b\sin\theta)^2}{b^2}\\ &= \cos^2\theta+\sin^2\theta\\ &= 1\\ &= \text{RHS}\end{aligned}$$

$\therefore P$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Question 5 (a) (iii)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}.\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{-xb^2}{ya^2}$$

Substitute $x=a\cos\theta,\ y=b\sin\theta$

$$\frac{dy}{dx}=\frac{-a\cos\theta.b^2}{b\sin\theta.a^2}=\frac{-b\cos\theta}{a\sin\theta}$$

$$y-y_1=m(x-x_1)$$

$$y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}.(x-a\cos\theta)$$

$$a\sin\theta y-ab\sin^2\theta=-xb\cos\theta+ab\cos^2\theta$$

$$xb\cos\theta+a\sin\theta y=ab(\sin^2\theta+\cos^2\theta)=ab$$

$$\therefore \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Question 5 (a) (iv)

Tangent to circle C_1 at A:

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{a} = 1$$

This cuts the x-axis when $y = 0$

$$\therefore \frac{x\cos\theta}{a} + 0 = 1$$

$$x = \frac{a}{\cos\theta}$$

Similarly, the tangent to the ellipse at P cuts the x-axis when $y = 0$

$$\therefore \frac{x\cos\theta}{a} + 0 = 1$$

$$\therefore x = \frac{a}{\cos\theta}$$

hence two tangents intersect at $\left(\frac{a}{\cos\theta}, 0\right)$.

Question 5 (b)

$$\frac{d}{dy}\ln\left(\frac{y}{1-y}\right) + c$$

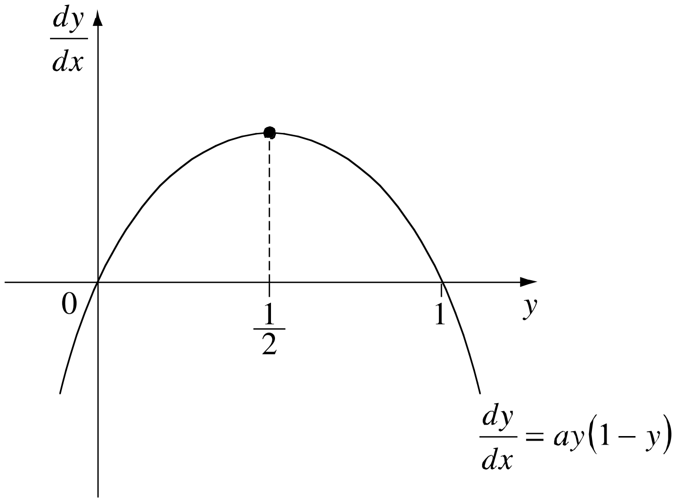
$$= \frac{d}{dy}\ln y - \frac{d}{dy}\ln(1-y) + \frac{d}{dy}c$$

$$= \frac{1}{y} + \frac{1}{1-y}$$

$$= \frac{1-y+y}{y(1-y)}$$

$$= \frac{1}{y(1-y)}$$

Question 5 (c) (i)



$\frac{dy}{dx}$ has a maximum value when $y = \frac{1}{2}$

Question 5 (c) (ii)

$$\frac{dy}{dx} = ay(1-y)$$

$$\frac{dx}{dy} = \frac{1}{ay(1-y)}$$

$$a\frac{dx}{dy} = \frac{1}{y(1-y)}$$

$$ax = \int \frac{dy}{y(1-y)}$$

$$ax = \ln\left(\frac{y}{1-y}\right) + c$$

$$\ln\frac{y}{1-y} = ax - c$$

$$\frac{y}{1-y} = Ae^{ax} \qquad (\text{where } A = e^{-c})$$

$$y = Ae^{ax}(1-y)$$

$$y = Ae^{ax} - yAe^{ax}$$

$$y\left(1 + Ae^{ax}\right) = Ae^{ax}$$

$$y = \frac{Ae^{ax}}{1 + Ae^{ax}}$$

$$\therefore y = \frac{1}{ke^{-ax} + 1} \qquad \left(\text{where } k = \frac{1}{A}\right)$$

Question 5 (c) (iii)

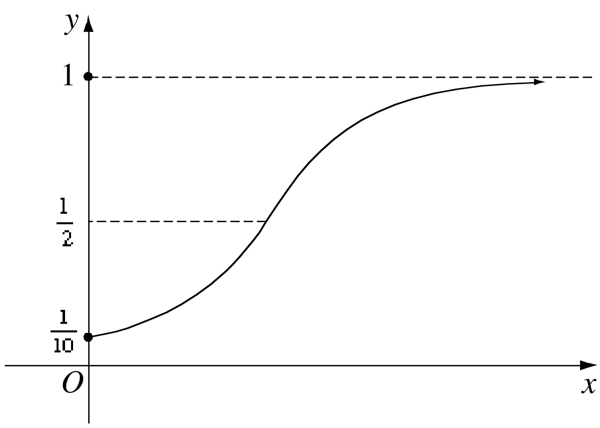
$$\frac{1}{10} = y(0) = \frac{1}{k+1}$$
$$\therefore k = 9$$

Question 5 (c) (iv)

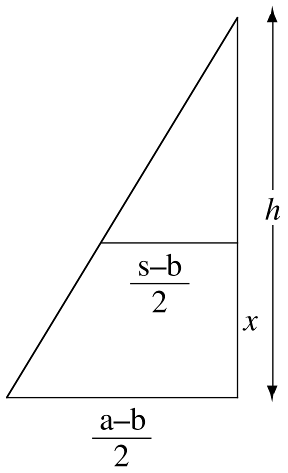
Part (c) (i) tells us that the curve is steepest at $y = \frac{1}{2}$.

There is a point of inflexion at $y = \frac{1}{2}$.

Question 5 (c) (v)



Question 6 (a) (i)



$$\frac{\frac{s-b}{2}}{\frac{a-b}{2}} = \frac{h-x}{h}$$

$$\frac{s-b}{a-b} = \frac{h-x}{h} = 1 - \frac{x}{h}$$

$$\therefore s-b = (a-b)\left(1-\frac{x}{h}\right) = a-b-(a-b)\frac{x}{h}$$

$$\therefore s = a-(a-b)\frac{x}{h}$$

Question 6 (a) (ii)

Volume of one slice $= s^2.\delta x$

Volume of solid $= \int_0^h s^2 \, dx$

$$= \int_0^h \left(a - \frac{(a-b)}{h} x \right)^2 dx$$

$$= \int_0^h \left(a^2 - \frac{2a(a-b)}{h} x + \frac{(a-b)^2}{h^2} x^2 \right)$$

$$= \left[a^2 x - \frac{a(a-b)}{h} x^2 + \frac{(a-b)^2}{h^2} \frac{x^3}{3} \right]_0^h$$

$$= a^2 h - a(a-b)h + \frac{(a-b)^2}{3} h$$

$$= h\left(a^2 - a^2 + ab + \frac{a^2 - 2ab + b^2}{3} \right)$$

$$= \frac{h}{3}\left(a^2 + ab + b^2 \right)$$

Question 6 (b)

Start of induction:

$n = 0: \quad \left(1+\sqrt{2}\right)^0 + \left(1-\sqrt{2}\right)^0 = 1 + 1 = 2$

$n = 1: \quad \left(1+\sqrt{2}\right)^1 + \left(1-\sqrt{2}\right)^1 = 2$

Induction step:
Induction assumption:

$$a_k = \left(1+\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^k$$

and $a_{k-1} = \left(1+\sqrt{2}\right)^{k-1} + \left(1-\sqrt{2}\right)^{k-1}$

Use the recursion relation:

$$\begin{aligned} a_{k+1} &= 2a_k + a_{k-1} \\ &= 2\left(\left(1+\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^k\right) + \left(1+\sqrt{2}\right)^{k-1} + \left(1-\sqrt{2}\right)^{k-1} \\ &= \left(1+\sqrt{2}\right)^{k-1}\left(1+2\sqrt{2}+2\right) + \left(1-\sqrt{2}\right)^{k-1}\left(1-2\sqrt{2}+2\right) \\ &= \left(1+\sqrt{2}\right)^{k+1} + \left(1-\sqrt{2}\right)^{k+1} \end{aligned}$$

where we use

$\left(1+\sqrt{2}\right)^2 = 1+2\sqrt{2}+2$ and

$\left(1-\sqrt{2}\right)^2 = 1-2\sqrt{2}+2$

Question 6 (c) (i)

$$\begin{aligned} &(\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + i^5\sin^5\theta \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \end{aligned}$$

Question 6 (c) (ii)

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

equating imaginary parts,

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2.\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta(1-2\sin^2\theta+\sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

Question 6 (c) (iii)

$$16\sin^5\frac{\pi}{10} - 20\sin^3\frac{\pi}{10} + 5\sin\frac{\pi}{10} = \sin\left(5\times\frac{\pi}{10}\right)$$
$$= \sin\frac{\pi}{2} = 1$$

$\therefore \sin\frac{\pi}{10}$ is a solution to $16x^5 - 20x^3 + 5x - 1 = 0$

Question 6 (c) (iv)

$$16x^5 - 20x^3 + 5x - 1 = (x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1)$$

Question 6 (c) (v)

$$16x^4 + 16x^3 - 4x^2 - 4x + 1$$
$$= (4x^2 + ax - 1)^2$$
$$= 16x^4 + 8ax^3 + \ldots$$
$$\therefore a = 2$$

hence $q(x) = 4x^2 + 2x - 1$

Question 6 (c) (vi)

$16x^5 - 20x^3 + 5x - 1 = (x-1)(4x^2 + 2x - 1)^2 = 0$

Solutions are $x = 1$ and $x = \dfrac{-2 \pm \sqrt{20}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$

hence the value of $\sin\dfrac{\pi}{10} = \dfrac{-1+\sqrt{5}}{4}$

Question 7 (a) (i)

In ΔADB and ΔKDC

$\angle ADB = \angle ADK + \angle KDB = \angle CDB + \angle KDB = \angle KDC$ (given)

$\angle ABD = \angle KCD$ (angles in the same segment)

$\therefore \Delta ADB \parallel\!| \Delta KDC$ (equiangular)

Question 7 (a) (ii)

In ΔADK and ΔBDC

$\dfrac{AK}{AD} = \dfrac{BC}{BD}$

$\therefore AK \times BD = AD \times BC$ ....①

In ΔADB and ΔKDC

$\dfrac{KC}{CD} = \dfrac{AB}{BD}$

$\therefore KC \times BD = CD \times AB$ ....②

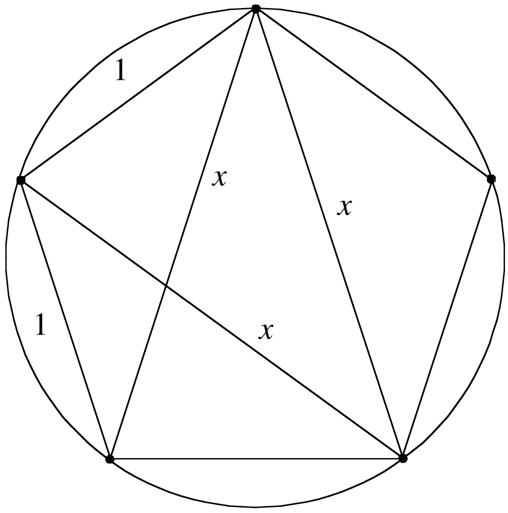
adding ① and ②

$AK \times BD + KC \times BD = AD \times BC + CD \times AB$

$BD(AK + KC) = AD \times BC + AB \times DC$

$\therefore BD \times AC = AD \times BC + AB \times DC$

Question 7 (a) (iii)



By part (ii)

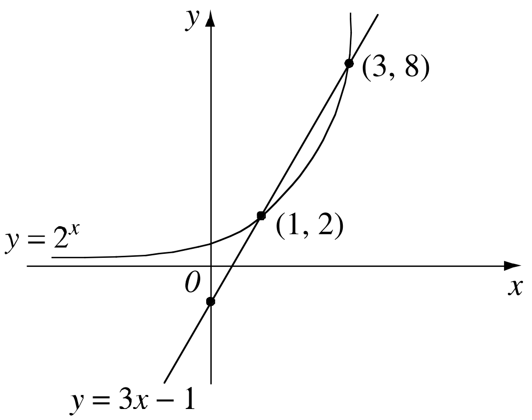
$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$\therefore x = \frac{1 \pm \sqrt{5}}{2}$$

since $x > 0$, $\quad x = \dfrac{1 + \sqrt{5}}{2}$

Question 7 (b)



for $x > 3$, $y = 2^x$ is greater than $y = 3x - 1$

$\therefore 2^x \geq 3x - 1$ for $x \geq 3$

Question 7 (c) (i)

$P'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2}$

$P'(x) = n(n-1)x^{n-2}(x-1)$

$P'(x) = 0$ when $x = 0$ or $x = 1$

Hence there are exactly two turning points.

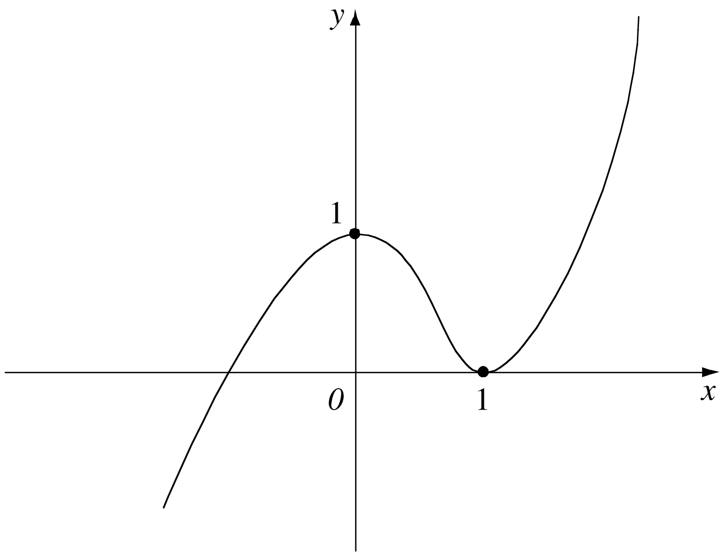
Question 7 (c) (ii)

$P(1) = (n-1) \times 1^n - n \times 1^{n-1} + 1 = 0$
$P'(1) = 0$ (shown in (i))
$\therefore P(x)$ has a double root when $x = 1$.

$$\left(\text{Note: } \begin{array}{l} P''(x) = n(n-1)^2 x^{n-2} - n(n-1)(n-2)x^{n-3} \\ P''(1) \neq 0 \quad \therefore P(x) \text{ does not have a triple root at } x = 1 \end{array} \right)$$

Question 7 (c) (iii)

$P(0) = 1$ and $P(1) = 0$
$P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$
(since the degree of $P(x)$ is odd)



The curve cuts the x-axis in only one place (other than $x = 1$).
$\therefore$ there is exactly one real zero of $P(x)$ other than 1.

Question 7 (c) (iv)

$$
\begin{aligned}
P(-1) &= (n-1)\times(-1)^n - n\times(-1)^{n-1} + 1 \\
&= (n-1)\times(-1) - n\times 1 + 1 (\text{since } n \text{ is odd}) \\
&= -n + 1 - n + 1 = -2n + 2
\end{aligned}
$$

$$
\begin{aligned}
P\left(-\frac{1}{2}\right) &= (n-1)\left(-\frac{1}{2}\right)^n - n\left(-\frac{1}{2}\right)^{n-1} + 1 \\
&= \frac{1}{2^n}\left\{-(n-1) - 2n + 2^n\right\} \\
&= \frac{1}{2^n}\left\{2^n - 3n + 1\right\} \\
&> 0 \text{ since } \frac{1}{2^n} > 0 \text{ and } 2^n - 3n + 1 > 0 \text{ (from(b))}
\end{aligned}
$$

Hence $-1<\alpha<-\frac{1}{2}$ because $P(x)$ changes sign between $x = -\frac{1}{2}$ and $x = -1$.

Question 7 (c) (v)

Since the coefficients are real, $4x^5 - 5x^4 + 1$ has zeros $1, 1, \alpha, \beta, \overline{\beta}$ where β is not real. Clearly, 1 and α have modulus less than 1.

$$
\begin{aligned}
\text{Product of roots} &= -\frac{1}{4} \\
\alpha\beta\overline{\beta} &= -\frac{1}{4} \\
\left|\alpha\,\beta\,\overline{\beta}\right| &= \left|-\frac{1}{4}\right| = \frac{1}{4} \\
|\alpha|\ |\beta|^2 &= \frac{1}{4} \\
\text{since } |\alpha| > \frac{1}{2},\ & |\beta| < 1
\end{aligned}
$$

Question 8 (a)

Using integration by parts

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \cos^{2n-1} x \, dx$$

$$= \sin x \cos^{2n-1} x \Big|_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{2n-2} x \, dx$$

$$= (2n-1)\int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\cos^{2n-2} x \, dx$$

$$= (2n-1)A_{n-1} - (2n-1)A_n$$

Hence

$$\begin{aligned}(2n-1)A_{n-1} &= A_n + (2n-1)A_n \\ &= 2n\, A_n\end{aligned}$$

So

$$nA_n = \frac{2n-1}{2}A_{n-1}$$

Question 8 (b)

Using integration by parts

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$$

$$= \int_0^{\frac{\pi}{2}} 1.\cos^{2n} x \, dx$$

$$= x\cos^{2n} x \Big|_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx$$

$$= 2n\int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx$$

Question 8 (c)

Using integration by parts on the integral in (b):

$$2\int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx$$

$$= x^2 \sin x \cos^{2n-1} x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x^2 \left(\sin x \cos^{2n-1} x\right)' dx$$

$$= -\int_0^{\frac{\pi}{2}} x^2 \left(\cos^{2n} x - (2n-1)\sin^2 x \cos^{2n-2} x\right) dx$$

$$= -B_n + (2n-1)\int_0^{\frac{\pi}{2}} x^2 \left(1 - \cos^2 x\right)\cos^{2n-2} x \, dx$$

$$= -B_n + (2n-1)\left(B_{n-1} - B_n\right)$$

$$= (2n-1)B_{n-1} - 2nB_n$$

Now use (b) to get

$$A_n = 2n\int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx$$

$$= (2n-1)nB_{n-1} - 2n^2 B_n$$

Dividing by n^2

$$\frac{A_n}{n^2} = \frac{2n-1}{n} B_{n-1} - 2B_n$$

Question 8 (d)

Dividing the identity in (c) by A_n and then using (a) we get

$$\frac{1}{n^2} = \frac{2n-1}{nA_n}B_{n-1} - 2\frac{B_n}{A_n}$$

$$= \frac{2n-1}{\frac{2n-1}{2}A_{n-1}}B_{n-1} - 2\frac{B_n}{A_n}$$

$$= 2\frac{B_{n-1}}{A_{n-1}} - 2\frac{B_n}{A_n}$$

Question 8 (e)

From (d) we get a telescoping sum:

$$\sum_{k=1}^{n}\frac{1}{k^2} = 2\sum_{k=1}^{n}\left(\frac{B_{k-1}}{A_{k-1}} - \frac{B_k}{A_k}\right)$$

$$= 2\frac{B_0}{A_0} - 2\frac{B_n}{A_n}$$

Now

$$A_0 = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}$$

$$B_0 = \int_0^{\frac{\pi}{2}} x^2\,dx = \frac{1}{3}x^3\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}\,\frac{\pi^3}{8}$$

Hence

$$2\frac{B_0}{A_0} = 2\,\frac{\frac{1}{3}\,\frac{\pi^3}{8}}{\frac{\pi}{2}} = \frac{\pi^2}{6}$$

and so

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$$

Question 8 (f)

Since $\cos^2 x = 1 - \sin^2 x \le 1 - \dfrac{4}{\pi^2}x^2$

for $0 \le x \le \dfrac{\pi}{2}$ we get

$$B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x\, dx \le \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4}{\pi^2}x^2\right)^n dx$$

Question 8 (g)

Using integration by parts

$$\int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4}{\pi^2}x^2\right)^n dx = \int_0^{\frac{\pi}{2}} x.x.\left(1 - \frac{4}{\pi^2}x^2\right)^n dx$$

$$= -\frac{\pi^2 x}{8(n+1)}\left(1 - \frac{4}{\pi^2}x^2\right)^{n+1}\Bigg|_0^{\frac{\pi}{2}} + \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}} \left(1 - \frac{4}{\pi^2}x^2\right)^{n+1} dx$$

$$= -\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}} \left(1 - \frac{4}{\pi^2}x^2\right)^{n+1} dx$$

Question 8 (h)

Note:

$dx = \frac{\pi}{2}\cos t\,dt$ and

if $x = 0$, then $t = 0$

if $x = \frac{\pi}{2}$, then $t = \frac{\pi}{2}$

Hence

$$\int_0^{\frac{\pi}{2}} \left(1 - \frac{4}{\pi^2}x^2\right)^{n+1} dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(1 - \sin^2 t\right)^{n+1} \cos t\,dt$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos^{2n+3} t\,dt$$

Using the information given

$$B_n \le \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}} \cos^{2n+3} t\,dt$$

Since $0 \le \cos t \le 1$ for $0 \le t \le \frac{\pi}{2}$ we get

$$B_n \le \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}} \cos^{2n} t\,dt = \frac{\pi^3}{16(n+1)}A_n$$

Question 8 (i)

Since $A_n, B_n > 0$ we get from part (e)

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n} < \frac{\pi^2}{6}$$

On the other hand, from (h)

$$2\frac{B_n}{A_n} \le \frac{\pi^3}{8(n+1)}, \text{ so}$$

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \le \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6}$$

Question 8 (j)

From (i) the limit is $\frac{\pi^2}{6}$ since

$$\frac{\pi^3}{8(n+1)} \to 0 \text{ as } n \to \infty.$$