BOARD OF STUDIES
NEW SOUTH WALES

# 2010 HSC Mathematics
# Sample Answers

This document contains 'sample answers', or, in the case of some questions, 'answers could include'. These are developed by the examination committee for two purposes. The committee does this:

(a) as part of the development of the examination paper to ensure the questions will effectively assess students' knowledge and skills, and

(b) in order to provide some advice to the Supervisor of Marking about the nature and scope of the responses expected of students.

The 'sample answers' or similar advice are not intended to be exemplary or even complete answers or responses. As they are part of the examination committee's 'working document', they may contain typographical errors, omissions, or only some of the possible correct answers.

**Question 1 (a)**

$$x^2 = 4x$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0 \qquad x = 0 \text{ or } x = 4$$

**Question 1 (b)**

$$\frac{1}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{\sqrt{5}+2}{5-4} = \sqrt{5}+2$$

$\therefore a = 2 \qquad b = 1$

**Question 1 (c)**

$$(x+1)^2 + (y-2)^2 = 25$$

**Question 1 (d)**

$2x + 3 = \pm 9$

$$2x+3 = 9 \quad \text{or} \quad 2x+3 = -9$$
$$2x = 6 \qquad\qquad 2x = -12$$
$$x = 3 \quad \text{or} \quad x = -6$$

**Question 1 (e)**

Let $y = x^2 \tan x$

$y' = x^2 \sec^2 x + 2x \tan x$

**Question 1 (f)**

$a = 1 \quad r = -\frac{1}{3} \qquad S = \frac{a}{1-r} = \frac{1}{1 - -\frac{1}{3}} = \frac{3}{4}$

**Question 1 (g)**

$x \geq 8$

**Question 2 (a)**

Let $y = \dfrac{\cos x}{x}$

$$y' = \frac{x(-\sin x) - \cos x}{x^2} = \frac{-x\sin x - \cos x}{x^2}$$

**Question 2 (b)**

$$(x-4)(x+3) < 0$$

$$-3 < x < 4$$

**Question 2 (c)**

$$y = \ln 3x$$

$$y' = \frac{1}{x}$$

When $x = 2$ $y' = \dfrac{1}{2}$

**Question 2 (d) (i)**

$$\int (5x+1)^{\frac{1}{2}}\,dx = \frac{1}{5}\cdot\frac{2}{3}(5x+1)^{\frac{3}{2}}$$

$$= \frac{2}{15}(5x+1)^{\frac{3}{2}} + C$$

**Question 2 (d) (ii)**

$$\int \frac{x}{4+x^2}\,dx = \frac{1}{2}\ln(4+x^2) + C$$

**Question 2 (e)**

$$\int_0^6 (x+k)\,dx = \left[\frac{x^2}{2} + kx\right]_0^6 = 18 + 6k$$

$$18 + 6k = 30$$
$$6k = 12$$
$$k = 2$$

**Question 3 (a) (i)**

$$\left(\frac{-2+12}{2}\ ,\ \frac{-4+6}{2}\right) = (5, 1)$$

**Question 3 (a) (ii)**

$$m_{BC} = \frac{8-6}{6-12} = -\frac{1}{3}$$

**Question 3 (a) (iii)**

$$m_{MN} = \frac{2-1}{2-5} = -\frac{1}{3}$$

$\therefore BC \parallel MN$

$\therefore \angle ANM = \angle ACB$ (corresponding angles on parallel lines equal)

and $\angle AMN = \angle ABC$ (corresponding angles on parallel lines equal)

(also $\angle A$ is common)

$\therefore \Delta ABC\ |||\ \Delta AMN$ (equiangular)

**Question 3 (a) (iv)**

$$y - 2 = -\frac{1}{3}(x-2)$$

or $x + 3y - 8 = 0$

**Question 3 (a) (v)**

$$\sqrt{(12-6)^2 + (6-8)^2} = \sqrt{36+4} = \sqrt{40}$$
$$= 2\sqrt{10}$$

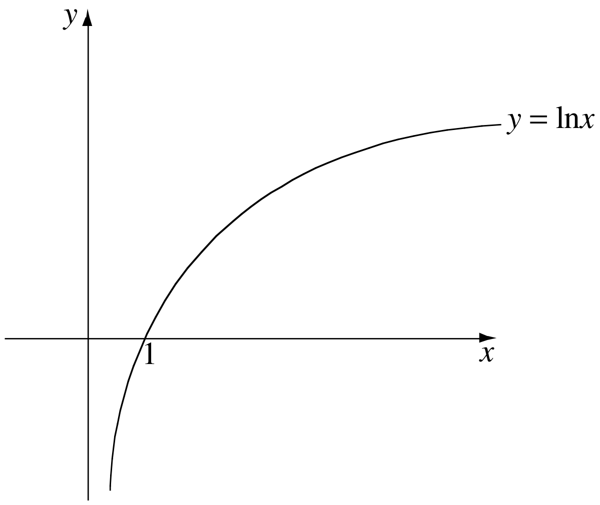
**Question 3 (a) (vi)**

Let $d$ = distance from $A$ to $BC$

$$\frac{1}{2} \times d \times 2\sqrt{10} = 44$$

$$\therefore \; d = \frac{44}{\sqrt{10}}$$

**Question 3 (b) (i)**




**Question 3 (b) (ii)**

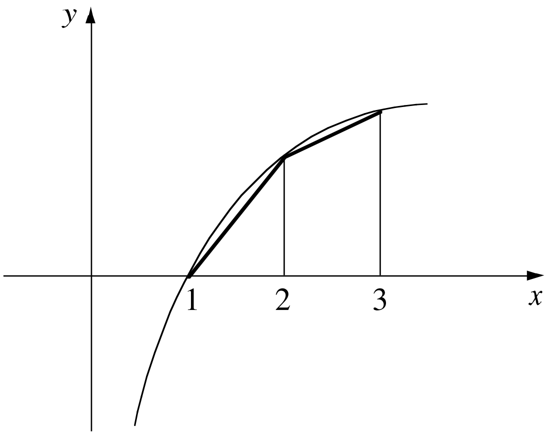
| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $y$ | 0 | $\ln 2$ | $\ln 3$ |

$$\int_1^3 \ln x \, dx \doteqdot \frac{1}{2}\left[0 + 2\ln 2 + \ln 3\right] = \frac{1}{2}\ln 12$$

$$= 1.24 \quad (2 \text{ dp})$$

**Question 3 (b) (iii)**

Approximation is less than the exact value since $y = \ln x$ is concave downwards and so trapezia lie beneath the curve.




**Question 4 (a) (i)**

Arithmetic series $a = 1$, $d = 0.75$

$T_9 = 1 + 8 \times .75$

$= 7$

She runs 7 km

**Question 4 (a) (ii)**

$10 = 1 + (n - 1) \times .75$

$n = 13$

$\therefore$ 13th week

**Question 4 (a) (iii)**

She runs 10 km in weeks 14, 15, …26

$\therefore$ Total $= \frac{n}{2}(a + \ell) + 10 \times 13$

$= \frac{13}{2}(1 + 10) + 130$

$= 201.5$ km

**Question 4 (b)**

$$\text{Area} = \int_0^2 \left(e^{2x} - e^{-x}\right) dx$$

$$= \left[\frac{1}{2}e^{2x} + e^{-x}\right]_0^2$$

$$= \left(\frac{1}{2}e^4 + e^{-2}\right) - \left(\frac{1}{2} + 1\right)$$

$$= \frac{1}{2}e^4 + e^{-2} - \frac{3}{2} \quad \text{sq units}$$

**Question 4 (c) (i)**

$$\frac{4}{12} \times \frac{3}{11} = \frac{1}{11}$$

**Question 4 (c) (ii)**

$$3 \times \frac{1}{11} = \frac{3}{11}$$

**Question 4 (c) (iii)**

$$1 - \frac{3}{11} = \frac{8}{11}$$

**Question 4 (d)**

$$\text{LHS} = f(x).f(-x)$$

$$= \left(1 + e^x\right)\left(1 + e^{-x}\right)$$

$$= 1 + e^{-x} + e^x + 1$$

$$= 2 + e^x + e^{-x}$$

$$\text{RHS} = f(x) + f(-x)$$

$$= 1 + e^x + 1 + e^{-x}$$

$$= \text{LHS}$$

**Question 5 (a) (i)**

$\pi r^2 h = 10$

$\therefore \quad h = \dfrac{10}{\pi r^2}$

$\therefore \quad A = 2\pi r^2 + 2\pi r\left(\dfrac{10}{\pi r^2}\right)$

$\quad\quad = 2\pi r^2 + \dfrac{20}{r}$

**Question 5 (a) (ii)**

$\dfrac{dA}{dr} = 4\pi r - 20r^{-2}$

$\dfrac{dA}{dr} = 0 \Rightarrow 4\pi r = \dfrac{20}{r^2}$

$\therefore r^3 = \dfrac{5}{\pi}$

$\quad r = \left(\dfrac{5}{\pi}\right)^{\frac{1}{3}} = 1.1675 \quad (4 \text{ dp})$

$\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{40}{r^3} > 0$ when $r$ is positive

$\therefore A$ is a minimum

**Question 5 (b) (i)**

$\text{LHS} = \sec^2 x + \sec x \tan x$

$\quad\quad = \dfrac{1}{\cos^2 x} + \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x}$

$\quad\quad = \dfrac{1 + \sin x}{\cos^2 x} = \text{RHS}$

**Question 5 (b) (ii)**

$\text{LHS} = \dfrac{1 + \sin x}{\cos^2 x} = \dfrac{1 + \sin x}{1 - \sin^2 x} = \dfrac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$

$\quad\quad = \dfrac{1}{1 - \sin x} = \text{RHS}$

**Question 5 (b) (iii)**

$$\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\,dx = \int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$$

$$= \Big[\ \tan x\ \Big]_0^{\frac{\pi}{4}} + \Big[\ \sec x\ \Big]_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \tan 0 + \sec\frac{\pi}{4} - \sec 0$$

$$= 1 - 0 + \sqrt{2} - 1$$

$$= \sqrt{2}$$

**Question 5 (c)**

$$A_1 = \int_a^1 \frac{1}{x}\,dx = \ln 1 - \ln a = -\ln a$$

$$A_1 = 1 \quad \therefore \ln a = -1$$

$$a = e^{-1}$$

$$A_2 = \int_1^b \frac{1}{x}\,dx = \ln b - \ln 1 = \ln b$$

$$\therefore \ln b = 1 \quad \therefore b = e$$

**Question 6 (a) (i)**

$f'(x) = (x+2)2x + (x^2+4)1$

$= 3x^2 + 4x + 4$

$\Delta = 4^2 - 4 \times 3 \times 4 < 0$

$\therefore$ the equation $f'(x) = 0$ has NO solutions.

$\therefore$ There are no turning points.

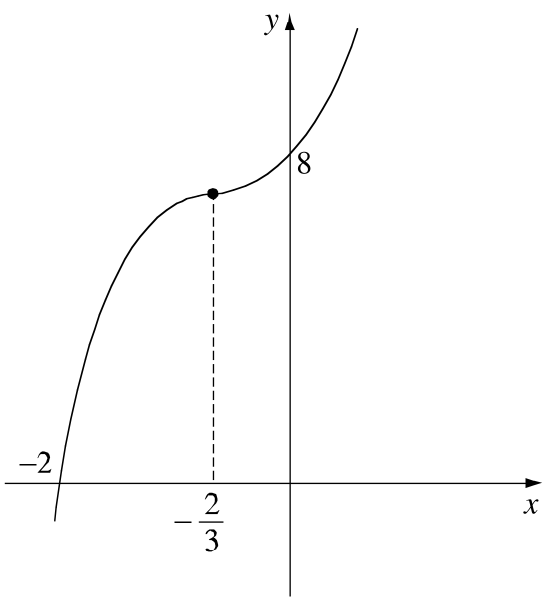
**Question 6 (a) (ii)**

$f''(x) = 6x + 4$

Concave down for $x < -\frac{2}{3}$

Concave up for $x > -\frac{2}{3}$

**Question 6 (a) (iii)**



**Question 6 (b) (i)**

Let $\angle POQ = \theta$

$$\ell = r\theta$$
$$\therefore\ 9 = 5\theta$$
$$\theta = \frac{9}{5}$$

**Question 6 (b) (ii)**

$OP = OQ$ (radii)

$\angle OPT = \angle OQT = 90^\circ$ (given)

$OT$ is common

$\therefore \Delta OPT \equiv \Delta OQT$ (RHS)

**Question 6 (b) (iii)**

From (ii) $\angle OPT = \frac{1}{2}\angle POQ = \frac{9}{10}$

$$\therefore \frac{PT}{5} = \tan\left(\frac{9}{10}\right)$$
$$\therefore\ PT = 5\tan\left(\frac{9}{10}\right)$$
$$\doteqdot 6.30 \text{ cm}$$

**Question 6 (b) (iv)**

$$\text{Area } POQT = 2 \times \frac{1}{2} \times 5 \times 5\tan\left(\frac{9}{10}\right)$$
$$= 25\tan\left(\frac{9}{10}\right)$$
$$\text{Area sector } OPQ = \frac{1}{2} \times 5^2 \times \frac{9}{5}$$
$$= \frac{45}{2}$$
$$\text{Shaded region} = 25\tan\left(\frac{9}{10}\right) - \frac{45}{2}$$
$$= 25(\tan 0.9 - 0.9)$$
$$\doteqdot 9.00 \text{ cm}^2$$

**Question 7 (a) (i)**

$\ddot{x} = 4\cos 2t$

$\dot{x} = 2\sin 2t + C_1$

When $t = 0 \quad \dot{x} = 1$

$\therefore 1 = 0 + C_1$

$\therefore \dot{x} = 2\sin 2t + 1$

**Question 7 (a) (ii)**

$$\dot{x} = 0 \quad \Rightarrow \quad \sin 2t = -\frac{1}{2}$$

$$\therefore 2t = \frac{7\pi}{6}$$

$$t = \frac{7\pi}{12}$$

**Question 7 (a) (iii)**

$x = -\cos 2t + t + C_2$

When $t = 0 \quad x = 0$

$\therefore 0 = -1 + 0 + C_2$

$\therefore C_2 = 1$

$\therefore x = 1 - \cos 2t + t$

**Question 7 (b) (i)**

$y' = 2x$ So the slope of tangent at $(-1, 1)$ is $-2$. So equation of tangent at $(-1, 1)$ is

$y - 1 = -2(x + 1)$

*ie* $\quad y = -2x - 1$

**Question 7 (b) (ii)**

At $C$, the tangent has slope the same as that of $AB$.

So, slope of tangent is $\dfrac{4-1}{2+1} = \dfrac{3}{3} = 1$.

So, as slope at $(x, x^2)$ is $2x$, at $C$, $2x = 1$. Hence $C$ is $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$.

$M$ is $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)\left(= \dfrac{A+B}{2}\right)$.

$C, M$ have same $x$-coordinate, so $CM$ is vertical.

**Question 7 (b) (iii)**

$x$-coordinate of $T$ is $\dfrac{1}{2}$. So $T$ is $\left(\dfrac{1}{2}, y\right)$, say. As $y = -2x - 1$,

we have $y = -2$. So $T$ is $\left(\dfrac{1}{2}, -2\right)$.

Slope of $BT = \dfrac{6}{\frac{3}{2}} = 4 =$ slope of tangent at $B$.

**Question 8 (a)**

$$P = Ae^{kt}$$

When $t = 0$ $P = 102$ $\therefore Ae^0 = 102$

$$\therefore P = 102e^{kt}$$

When $t = 75$ $P = 200\ 000\ 000 = 2 \times 10^8$

$$\therefore 2 \times 10^8 = 102e^{75k}$$

$$\therefore k = \frac{1}{75}\ln\left(\frac{2 \times 10^8}{102}\right) \doteqdot 0.19318$$

When $t = 100$

$$P = 102e^{100k}$$

$$\doteqdot 2.503 \times 10^{10}$$

**Question 8 (b)**

Let $p$ = probability that coin shows a head

$\therefore p^2 = 0.36$

$\therefore p = 0.6$

Probability that coin shows a tail $= 1 - p$

$= 0.4$

$\therefore$ Probability that both coins show a tail $= 0.4^2$

$= 0.16$

**Question 8 (c) (i)**

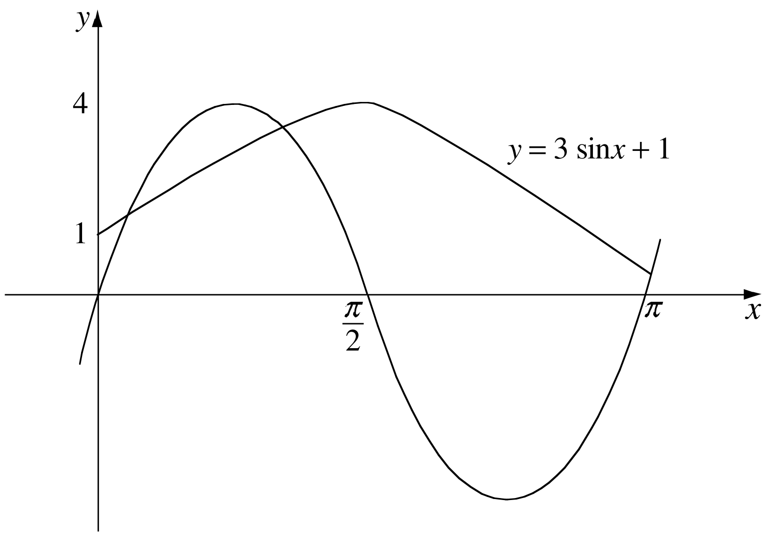
$A = 4$

**Question 8 (c) (ii)**

Period $= \pi = \dfrac{2\pi}{b}$

$\therefore b = 2$

**Question 8 (c) (iii)**




**Question 8 (d)**

$f'(x) = 3x^2 - 6x + k$

$f$ is increasing if $f'$ is always positive

$\therefore \Delta < 0$

$36 - 12k < 0$

$\therefore k > 3$

Note: If $k \geq 3$, $f$ is an increasing function, since if $x_1 < x_2$ then $f(x_1) < f(x_2)$.

**Question 9 (a) (i)**

$$P = 500(1.005)^{240} + 500(1.005)^{239} + \ldots + 500(1.005)^{1}$$
$$= 500\left[(1.005)^{1} + (1.005)^{2} + \ldots + (1.005)^{240}\right]$$
$$= 500(1.005)\left[\frac{(1.005)^{240} - 1}{1.005 - 1}\right]$$
$$= 232\ 175.55$$

**Question 9 (a) (ii) (1)**

$$A_0 = P$$
$$A_1 = P(1.005) - 2000$$
$$A_2 = A_1(1.005) - 2000$$
$$= P(1.005)^2 - 2000(1.005) - 2000$$

Generally $A_n = A_{n-1}(1.005) - 2000$

so that $A_n = P(1.005)^n - 2000(1.005)^{n-1} - \ldots - 2000$

$$= P(1.005)^n - 2000\left[1 + (1.005) + \ldots + (1.005)^{n-1}\right]$$
$$= P(1.005)^n - 2000\left[\frac{(1.005)^n - 1}{1.005 - 1}\right]$$
$$= P(1.005)^n - \frac{2000}{0.005}\left((1.005)^n - 1\right)$$
$$= P(1.005)^n - 400\ 000\left((1.005)^n - 1\right)$$
$$= P(1.005)^n - 400\ 000(1.005)^n + 400\ 000$$
$$= (P - 400\ 000)(1.005)^n + 400\ 000$$

**Question 9 (a) (ii) (2)**

Find $n$ so that $A_n = 0$

$$(1.005)^n = \frac{400\ 000}{400\ 000 - P}$$
$$n\log(1.005) = \log\left(\frac{400\ 000}{400\ 000 - 232\ 175.55}\right)$$
$$n \doteqdot 174.143$$

$\therefore$ There will be money in the account for 175 months.

**Question 9 (b) (i)**

$0 \le x < 2$

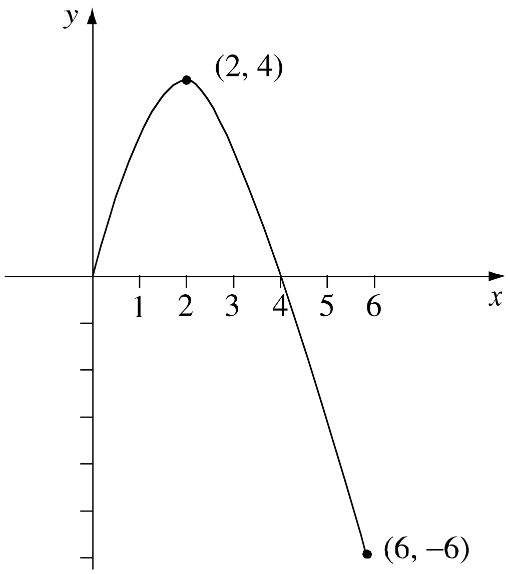
**Question 9 (b) (ii)**

4

**Question 9 (b) (iii)**

–6

**Question 9 (b) (iv)**



**Question 10 (a) (i)**

Let $\angle CAD = \alpha$
Then $\Delta ACD$ has two angles equal to $\alpha$ (since it is isosceles) and $\Delta ABC$ has two angles equal to $\alpha$ (since it is isosceles). Hence (since 3rd angles must be equal) $\Delta ABC \parallel\!\mid \Delta ACD$ (equiangular).

**Question 10 (a) (ii)**

Since corresponding sides are in proportion,

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{a+y}{x} = \frac{x}{a}$$

$$\therefore a(a+y) = x^2$$

$$\therefore x^2 = a^2 + ay$$

**Question 10 (a) (iii)**

$$x^2 = a^2 + a^2 - 2a^2\cos\theta \quad \text{(using cosine rule)}$$

$$\therefore a^2 + ay = 2a^2 - 2a^2\cos\theta$$

$$ay = a^2 - 2a^2\cos\theta$$

$$y = a - 2a\cos\theta$$

$$= a(1 - 2\cos\theta)$$

**Question 10 (a) (iv)**

$$2\cos\theta \ge -2$$

$$\therefore -2\cos\theta \le 2$$

$$\therefore 1 - 2\cos\theta \le 1 - -2 = 3$$

$$\therefore y \le 3a$$

**Question 10 (b) (i)**

$$\text{Volume} = \pi \int y^2\, dx$$

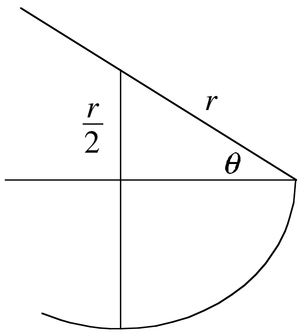
$$= \pi \int_{r\sin\theta}^{r} \left(r^2 - x^2\right) dx$$

$$= \pi \left[ r^2 x - \frac{x^3}{3} \right]_{r\sin\theta}^{r}$$

$$= \pi \left[ r^3 - \frac{r^3}{3} - r^3 \sin\theta + \frac{r^3 \sin^3\theta}{3} \right]$$

$$= \frac{\pi r^3}{3}\left[ 2 - 3\sin\theta + \sin^3\theta \right]$$

**Question 10 (b) (ii) (1)**




$$\sin\theta = \frac{\frac{r}{2}}{r} = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

**Question 10 (b) (ii) (2)**

$$\text{New volume} = \frac{\pi r^3}{3}\left( 2 - 3\sin\frac{\pi}{6} + \sin^3\frac{\pi}{6} \right)$$

$$= \frac{\pi r^3}{3}\left( 2 - \frac{3}{2} + \frac{1}{8} \right) = \left(\frac{5}{8}\right)\frac{\pi r^3}{3}$$

$$\text{Volume of hemisphere} = \frac{2\pi r^3}{3}$$

$$\therefore \text{fraction} = \frac{\frac{5}{8}}{2} = \frac{5}{16}$$