BOARD OF STUDIES
NEW SOUTH WALES

2011 Mathematics Extension 2
HSC Examination
'Sample Answers'

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This material is also provided to the Supervisor of Marking, to give some guidance about the nature and scope of the responses the committee expected students would produce. How sample answers are used at marking centres varies. Sample answers may be used extensively and even modified at the marking centre OR they may be considered only briefly at the beginning of marking. In a few cases, the sample answers may not be used at all at marking.

The Board publishes this information to assist in understanding how the marking guidelines were implemented.

The 'sample answers' or similar advice contained in this document are not intended to be exemplary or even complete answers or responses. As they are part of the examination committee's 'working document', they may contain typographical errors, omissions, or only some of the possible correct answers.

Question 1 (a)

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\,\frac{x^2}{2} + C$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Question 1 (b)

$$\int_0^3 x\sqrt{x+1}\,dx = \int_{0+1}^{3+1}(u-1)\sqrt{u}\,du$$

$$u = x+1$$
$$du = dx$$
$$x = u-1$$

$$= \int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4$$

$$= \frac{2}{5}\left(2^5-1\right) - \frac{2}{3}\left(2^3-1\right)$$

$$= \frac{2}{5}(32-1) - \frac{2}{3}(8-1)$$

$$= \frac{2}{5}\cdot 31 - \frac{2}{3}\cdot 7$$

$$= \frac{62}{5} - \frac{14}{3} = \frac{186-70}{15} = \frac{116}{15}$$

Question 1 (c) (i)

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$$
$$= \frac{ax(x-1) + b(x-1) + cx^2}{x^2(x-1)}$$

hence $1 = ax(x-1) + b(x-1) + cx^2$ for all real x

$x = 0 \quad \Rightarrow \quad b = -1$
$x = 1 \quad \Rightarrow \quad c = 1$

equating coefficients of $x^2 \quad \Rightarrow \quad 0 = a + c \quad \therefore a = -1$

hence $\dfrac{1}{x^2(x-1)} = -\dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{1}{x-1}$

Question 1 (c) (ii)

$$\int \frac{dx}{x^2(x-1)} = -\ln x + \frac{1}{x} + \ln(x-1) + C$$

(integrate each term in the identity from part (i))

Question 1 (d)

$$\cos^3\theta = \cos\theta\left(1 - \sin^2\theta\right)$$

hence

$$\int \cos^3\theta\, d\theta = \int \cos\theta\, d\theta - \int \cos\theta \sin^2\theta\, d\theta$$

so

$$\int \cos^3\theta\, d\theta = \sin\theta - \frac{1}{3}\sin^3\theta + C$$

Question 1 (e)

$$\int_{-1}^{1} \frac{1}{5 - 2t + t^2}\, dt = \int_{-1}^{1} \frac{1}{(t-1)^2 + 4}\, dt \qquad \text{(completion of square)}$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{t-1}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\tan^{-1}0 - \tan^{-1}(-1)\right)$$

$$= \frac{1}{2}\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

Question 2 (a) (i)

$$
\begin{aligned}
\overline{w} + z &= 2 + 3i + 3 + 4i \\
&= 5 + 7i
\end{aligned}
$$

Question 2 (a) (ii)

$$
\begin{aligned}
|w|^2 &= 2^2 + 3^2 \\
&= 13 \\
|w| &= \sqrt{13}
\end{aligned}
$$

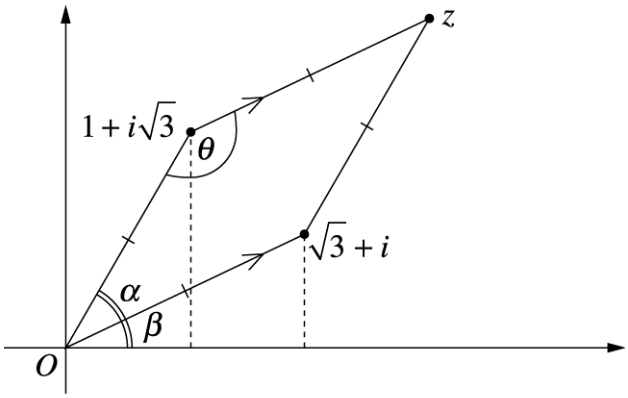
Question 2 (a) (iii)

$$
\begin{aligned}
\frac{w}{z} &= \frac{2-3i}{3+4i} \cdot \frac{3-4i}{3-4i} \\
&= \frac{6-8i-9i-12}{25} \\
&= -\frac{6}{25} - \frac{17}{25}i
\end{aligned}
$$

Question 2 (b) (i)

$$z = 1 + i\sqrt{3} + \sqrt{3} + i$$
$$= \left(1 + \sqrt{3}\right) + i\left(1 + \sqrt{3}\right)$$

Question 2 (b) (ii)



$$\tan\beta = \frac{1}{\sqrt{3}} \qquad \tan\alpha = \frac{\sqrt{3}}{1}$$
$$\beta = \frac{\pi}{6} \qquad = \sqrt{3}$$
$$\therefore \quad \alpha = \frac{\pi}{3}$$

$$\therefore \quad \alpha - \beta = \frac{\pi}{6}$$

$$\therefore \quad \theta = \pi - \frac{\pi}{6}$$
$$\therefore \quad \theta = \frac{5\pi}{6}$$

Question 2 (c)

Let $z = r(\cos\theta + i\sin\theta)$

by de Moivre's theorem,

$$\left(r(\cos\theta + i\sin\theta)\right)^3 = 8(\cos 2k\pi + i\sin 2k\pi), \qquad k \text{ an integer}$$

$$r^3(\cos 3\theta + i\sin 3\theta) = 2^3(\cos 2k\pi + i\sin 2k\pi)$$

equating moduli: $r = 2$

equating arguments: $3\theta = 2k\pi$

$$\theta = \frac{2k\pi}{3}$$

hence $z = 2\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), \qquad k$ an integer

$$= 2(\cos 0 + i\sin 0) \quad \text{or} \quad 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\text{or} \quad 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

(all equivalent solutions)

Question 2 (d) (i)

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2(i\sin\theta) + 3\cos(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

Question 2 (d) (ii)

by de Moivre's theorem, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

equating real parts:

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$$

so $\quad \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right) = \cos 3\theta$

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta$$

$$\cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$$

Question 2 (d) (iii)

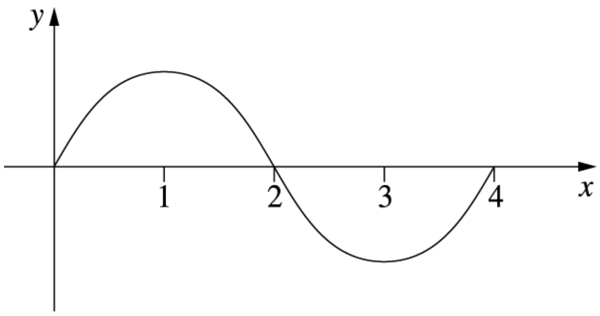
$4\cos^3\theta - 3\cos\theta = 1$

hence by part (ii)

$\cos 3\theta = 1$

The smallest positive solution is $\dfrac{2\pi}{3}$

Question 3 (a) (i)



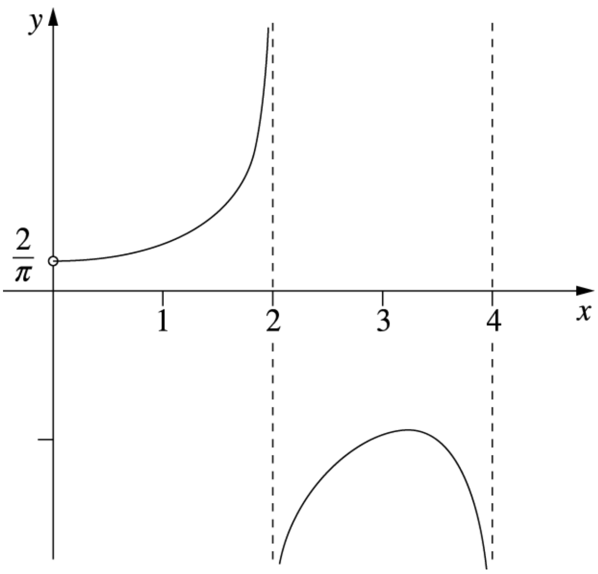
$$y = \sin\frac{\pi}{2}x$$

Question 3 (a) (ii)

$$\lim_{x\to 0}\frac{x}{\sin\frac{\pi}{2}x} = \frac{2}{\pi}\lim_{x\to 0}\frac{\frac{\pi}{2}x}{\sin\frac{\pi}{2}x}$$

$$= \frac{2}{\pi}$$

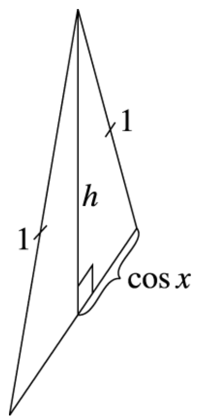
Question 3 (a) (iii)



$$y = \frac{x}{\sin\frac{\pi}{2}x}$$

Asymptotes $x = 2, x = 4$

Question 3 (b)



The height of the isosceles triangle is $\sin x$

The volume of a vertical slice is $\frac{1}{2} \cdot 2\cos x \sin x \triangle x$

The total volume is $\lim\limits_{\triangle x \to 0} \sum\limits_{\substack{\text{all}\\\text{slices}}} \cos x \sin x \triangle x$

$$= \int_0^{\frac{\pi}{2}} \cos x \sin x \, dx$$

$$= \left[\frac{1}{2}\sin^2 x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \text{ units}^3$$

Question 3 (c)

<u>Prove</u>: $(2n)! \geq 2^n (n!)^2$

For $n = 1$, it is
$2! \geq 2$, which is true.

So, assume it's true for n.

That is, assume
$$(2n)! \geq 2^n (n!)^2$$

then $\big(2(n+1)\big)!$

$$\geq (2n+2)(2n+1)(n!)^2 \, 2^n$$

$$\geq 2(n+1)(2n+1)(n!)^2 \, 2^n$$

$$\geq 2^{n+1}(n+1)(n+1)(n!)^2$$

$$= 2^{n+1}\big((n+1)!\big)^2$$

Question 3 (d) (i)

$$
\begin{aligned}
e^2 &= 1 + \frac{b^2}{a^2} \qquad\qquad a^2 = 16, \quad b^2 = 9 \\
&= 1 + \frac{9}{16} \\
&= \frac{25}{16} \\
e &= \frac{5}{4} \qquad\qquad (e > 0)
\end{aligned}
$$

Question 3 (d) (ii)

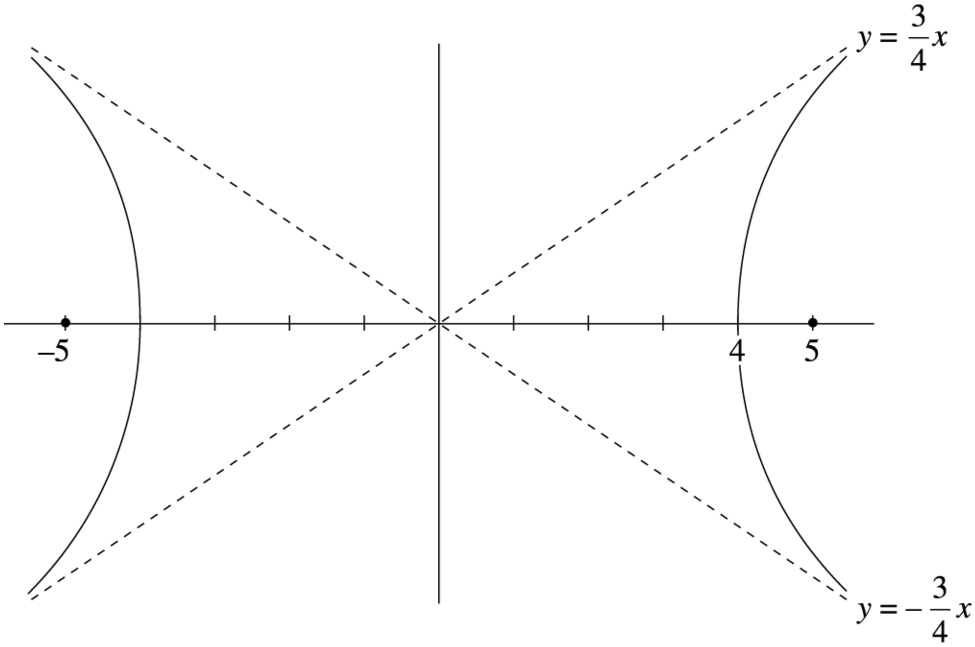
The foci are $(\pm ae, 0) = (\pm 5, 0)$

Question 3 (d) (iii)

$y = \pm \dfrac{b}{a} x$

The asymptotes are $y = \pm \dfrac{3}{4} x$

Question 3 (d) (iv)



Question 3 (d) (v)

$e^2 = 1 + \dfrac{b^2}{a^2}$ $\quad e \to \infty$ implies $\dfrac{b^2}{a^2} \to \infty$

hence the asymptotes approach a vertical line and the hyperbola straightens out.

Question 4 (a) (i)

$$\left|x+iy-a\right|^2 - \left|x+iy-b\right|^2 = 1$$

$$\left|(x-a)+iy\right|^2 - \left|(x-b)+iy\right|^2 = 1$$

$$(x-a)^2 + y^2 - \left((x-b)^2 + y^2\right) = 1$$

$$(x-a)^2 - (x-b)^2 = 1$$

$$\cancel{x^2} - 2ax + a^2 - \left(\cancel{x^2} - 2bx + b^2\right) = 1$$

$$x(2b-2a) + a^2 - b^2 = 1$$

$$x = \frac{1+b^2-a^2}{2(b-a)}$$

$$= \frac{1}{2(b-a)} + \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

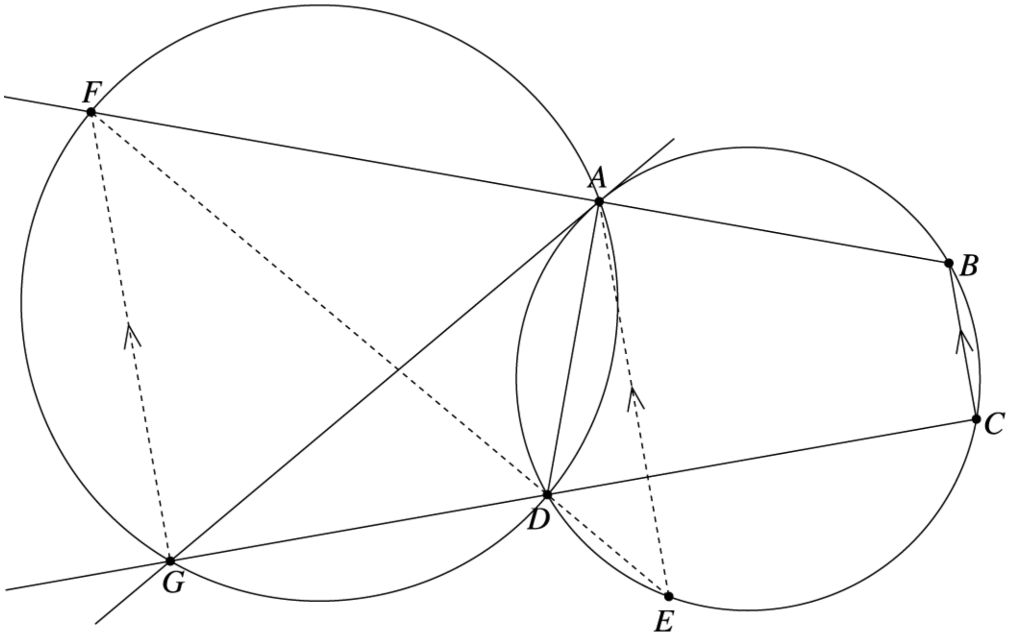
as required

Question 4 (a) (ii)

$\dfrac{a+b}{2} + \dfrac{1}{2(b-a)}$ is a constant,

hence the locus is a vertical line.

Question 4 (b)



Question 4 (b) (i)

$\angle GDH = \angle ABC$ (exterior opposite angle of cyclic quadrilateral $ABCD$)

Extend AF to H
$\angle GFH = \angle ABC$ (corresponding angles, $FG \,||\, BC$)

hence $\angle GDA = \angle GFH$
thus $FADG$ is cyclic (since the exterior opposite angle is equal).

Question 4 (b) (ii)

Alternate angles, $FG \,||\, AE$

Question 4 (b) (iii)

$\angle AED = \angle GFD$ (part (ii))
$= \angle GAD$ (angles at the circumference, standing on the arc GD of the circle through F, A, D and G)

hence GA is a tangent (since the angle in the alternative segment is equal to the angle between GA and the chord AD).

Question 4 (c) (i)

$y = Af(t) + Bg(t)$

$\dot{y} = A\dot{f}(t) + B\dot{g}(t)$

$\ddot{y} = A\ddot{f}(t) + B\ddot{g}(t)$

Substitute in the differential equations:

$\left(A\ddot{f}(t) + B\ddot{g}(t)\right) + 3\left(A\dot{f}(t) + B\dot{g}(t)\right) + \left(Af(t) + Bg(t)\right)$

$= \left(A\ddot{f}(t) + 3A\dot{f}(t) + 2A(t)\right) + \left(B\ddot{g}(t) + 3B\dot{g}(t) + 2Bg(t)\right)$

$= \quad 0 \quad + \quad 0$

since $f(t)$, $g(t)$ are solutions of the differential equation

$= 0$, as required.

Hence $Af(t) + Bg(t)$ is a solution.

Question 4 (c) (ii)

Substitute the trial from $y = e^{kt}$ into the differential equation:

$\dot{y} = ke^{kt} \qquad \ddot{y} = k^2e^{kt}$

$k^2e^{kt} + 3ke^{kt} + 2e^{kt} = 0$

$e^{kt}\left(k^2 + 3k + 2\right) = 0$

$e^{kt}(k+1)(k+2) = 0$

$k = -1$ or $k = -2 \quad \left(e^{kt} > 0\right)$

Question 4 (c) (iii)

$y = Ae^{-2t} + Be^{-t}$; $\qquad \dfrac{dy}{dt} = -2Ae^{-2t} - Be^{-t}$

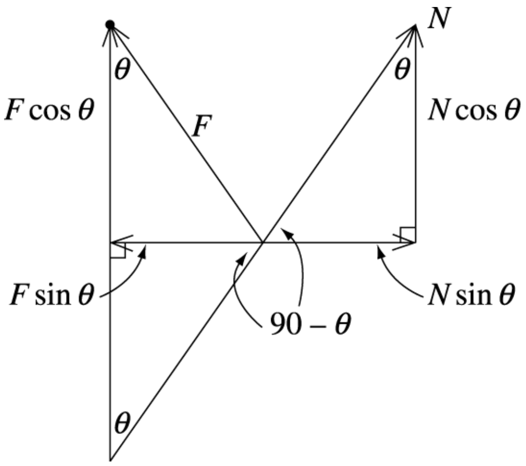
Using that $y = 0$ when $t = 0$:
$0 = A + B$ ①

Using that $\dfrac{dy}{dt} = 1$ when $t = 0$:
$1 = -2A - B$ ②

Adding ① + ② $\quad 1 = -A \quad \therefore A = -1$
hence $\quad B = 1$

thus $\quad y = e^{-t} - e^{-2t}$

Question 5 (a) (i)



Resolving horizontally: Net force $= m\omega^2 r$ (moves in a circle with uniform motion)

$F sin\theta - N\sin\theta = m\omega^2 r$ ①

Resolving vertically: Net force $= mg$ (gravitational force)

$F\cos\theta + N\cos\theta = mg$ ②

Question 5 (a) (ii)

From ①, $F - N = m\omega^2 r\operatorname{cosec}\theta$ ①′

From ②, $F + N = mg\sec\theta$ ②′

Subtracting ②′ – ①′; $2N = mg\sec\theta - m\omega^2 r\operatorname{cosec}\theta$

$$N = \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r\operatorname{cosec}\theta$$

Question 5 (a) (iii)

When in contact with sphere, the reaction force $N \geq 0$. That is if

$$\frac{1}{2}m\omega^2 r \operatorname{cosec}\theta \leq \frac{1}{2}mg\sec\theta$$

$$\omega^2 \leq \frac{g}{r}\frac{\sec\theta}{\operatorname{cosec}\theta}$$

$$= \frac{g}{r}\tan\theta \qquad \text{but } \tan\theta = \frac{r}{h}$$

$$\text{Thus } \omega^2 \leq \frac{g}{r}\frac{r}{h} = \frac{g}{h}$$

$$\omega^2 \leq \frac{g}{h}$$

$$\omega \leq \sqrt{\frac{g}{h}}$$

Question 5 (b)

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r}$$

$$= \frac{p(1+q)(1+r) + q(1+p)(1+r) - r(1+p)(1+q)}{(1+p)(1+q)(1+r)}$$

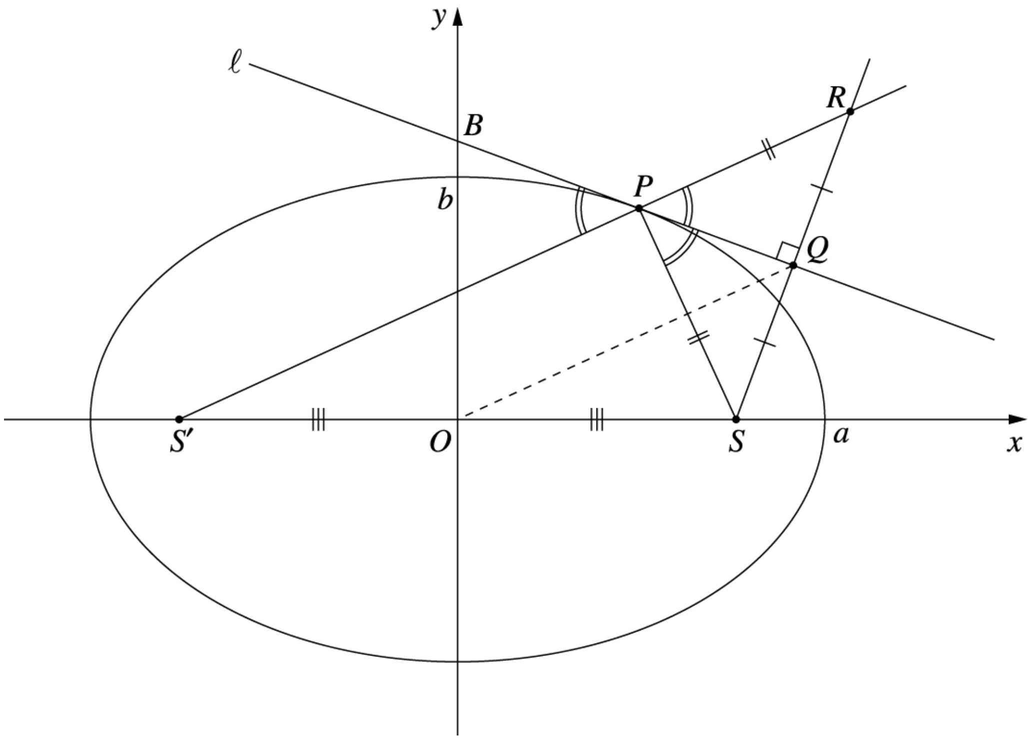
$$= \frac{p(1+q+r+qr) + q(1+p+r+pr) - r(1+p+q+pq)}{(1+p)(1+q)(1+r)}$$

$$= \frac{(p+q-r) + pq + \cancel{pr} + pqr + qp + \cancel{qr} + \cancel{pqr} - \cancel{rp} - \cancel{rq} - \cancel{rpq}}{(1+p)(1+q)(1+r)}$$

$$= \frac{(p+q-r) + pq(2+r)}{(1+p)(1+q)(1+r)}$$

$$\geq 0$$

Question 5 (c)



Question 5 (c) (i)

Let line ℓ make an intercept on the y-axis at B.
By the reflection property of an ellipse.

$$\angle S'PB = \angle SPQ$$

Now $\angle S'PB = \angle RPQ$ (vertically opposite angles)

in $\triangle RPQ$ and $\triangle SPQ$

- $\angle S'PB = \angle SPQ$ and $\angle S'PB = \angle RPQ$

  $\therefore\ \angle SPQ = \angle RPQ$

- $\angle RQP = \angle SQP = 90°$ (given $PQ \perp RS$)

- PQ is in common

  $\therefore \triangle RPQ \equiv \triangle SPQ$ (A A S)

$\therefore$ $SQ = QR$ (corresponding sides in congruent triangles)

Question 5 (c) (ii)

$\triangle SPR$ is isosceles $\triangle$ since $\triangle RPQ \equiv \triangle SPQ$

$\therefore SP = PR$

Now $S'P + PS = 2a$ (locus condition of an ellipse)

$\therefore S'P + PR = S'R = 2a$

Question 5 (c) (iii)

Join OQ where O is the midpoint of $S'S$ and Q is midpoint of SR.

Now, in $\triangle S'SR$ and $\triangle OSQ$,

$\frac{S'S}{OS} = \frac{RS}{QS} = \frac{2}{1}$ (since O and Q are midpoints of $S'S$ and RS respectively)

$\angle S'SR$ is in common

$\therefore \triangle SS'R \,|||\, \triangle OSQ$ (2 sides in proportion and included angle equal)

Now, $\frac{S'R}{OQ} = \frac{2}{1}$ (corresponding sides in similar triangles)

$\therefore \quad S'R = 2OQ$

but $S'R = 2a$

$\therefore \quad 2OQ = 2a$

$OQ = a$

$\therefore$ OQ is the radius of a circle with centre O and radius a units, which is given by $x^2 + y^2 = a^2$.

Question 6 (a) (i)

As the acceleration $\dot{v} \to 0$, the velocity tends to a limiting velocity, called the terminal velocity. From the equation of motion, this means

as $mg - kv^2 \to 0$ then $v \to v_T$

so $v_T^2 = \dfrac{mg}{k}$

$$v_T = \sqrt{\frac{mg}{k}}$$

Question 6 (a) (ii)

$$m\frac{dv}{dt} = mg - kv^2$$

$$\int_{v_0}^{v} \frac{m}{mg - kv^2}\, dv = \int_0^t dt$$

$$t = \frac{m}{k}\int_{v_0}^{v} \frac{dv}{\left(\frac{mg}{k}\right) - v^2} = \frac{v_T^2}{g}\int_{v_0}^{v} \frac{dv}{v_T^2 - v^2} \quad \text{as } \frac{m}{k} = \frac{v_T^2}{g}$$

If $v_o < v_T$, then $v < v_T$ all the time. So

$$t = \frac{v_T^2}{g}\int_{v_0}^{v} \frac{dv}{v_T^2 - v^2}$$

$$= \frac{v_T^2}{2gv_T}\int_{v_0}^{v} \frac{1}{v_T - v} + \frac{1}{v_T + v}\, dv$$

$$= \frac{v_T^2}{2gv_T}\Big\{\big(-\ln(v_T - v)\big) + \ln(v_T + v)\Big\}_{v_0}^{v}$$

$$= \frac{v_T^2}{2gv_T}\Big\{\ln(v_T + v) - \ln(v_T - v) - \ln(v_T + v_0) + \ln(v_T - v_0)\Big\}$$

$$= \frac{v_T}{2g}\left\{\ln\left[\frac{(v_T + v)(v_T - v_0)}{(v_T + v_0)(v_T - v)}\right]\right\} \qquad \text{as required.}$$

If $v_o > v_T$, then $v > v_T$ all the time. Then replace $v_T - v$ by $-(v - v_T)$ in the above calculation. This leads to the same result.

Question 6 (a) (iii)

When $v_0 = 3v_T$, $v = \frac{3}{2}v_T$ then for Gil:

$$t = \frac{v_T}{2g}\ln\frac{\frac{5}{2}v_T \times (-2)v_T}{4v_T \times \left(-\frac{1}{2}\right)v_T}$$

$$= \frac{v_T}{2g}\ln\frac{5}{2}$$ This is the time it takes for Gil’s speed to halve.

When $v_0 = \frac{1}{3}v_T$, $v = \frac{2}{3}v_T$ then for Jac:

$$t = \frac{v_T}{2g}\ln\frac{\frac{5}{3}v_T \times \frac{2}{3}v_T}{\frac{4}{3}v_T \times \frac{1}{3}v_T}$$

$$= \frac{v_T}{2g}\ln\frac{5}{2}$$ This is the time it takes for Jac’s speed to double.

Hence in the time when Jac’s speed has doubled, Gil’s speed has halved.

Question 6 (b) (i)

$y = \left(f(x)\right)^3$

$y' = 3f(x)^2 \times f'(x)$

There will be a stationary point if $y' = 0$, that is if

$3f(x)^2 f'(x) = 0$

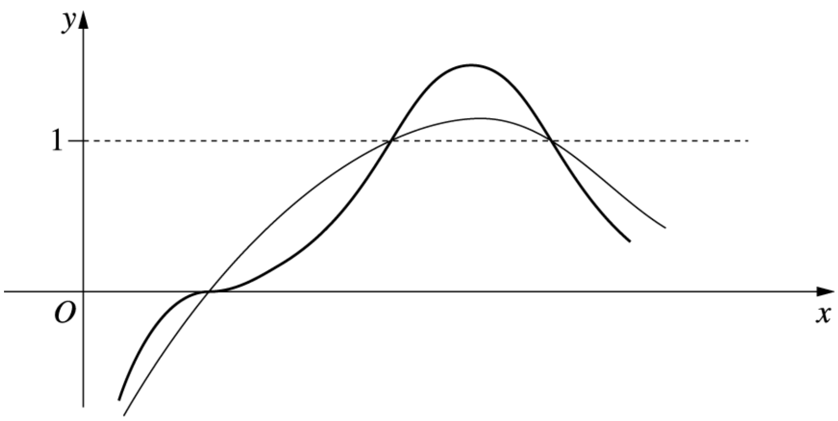
$f(x) = 0 \quad \text{or} \quad f'(x) = 0$

Hence if $f(a) = 0$ or $f'(a) = 0$ then there is a stationary point at $x = a$.

Question 6 (b) (ii)

By part (i) $y = \left(f(x)\right)^3$ has a stationary point at $x = a$. As $f'(a) \neq 0$, $f(x)$ is either strictly increasing or strictly decreasing near $x = a$. The same is true for $\left(f(x)\right)^3$, so there is a point of inflexion at $x = a$.

Question 6 (b) (iii)



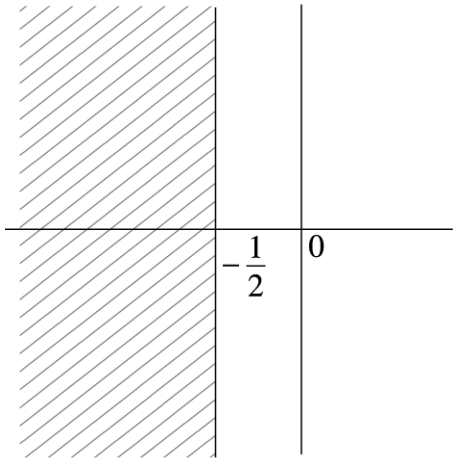
Question 6 (c)

Let $z = x + iy$. Then

$$\left|1 + \frac{1}{x+iy}\right| \le 1$$

$$\left|\frac{x+iy+1}{x+y}\right| \le 1$$

$$\left|(x+1)+iy\right| \le \left|x+iy\right|$$



Squaring:

$$(x+1)^2 + y^2 \le x^2 + y^2$$

$$2x + 1 \le 0$$

$$x \le -\frac{1}{2}$$

Alternative solution

Describe region in complex plane given by $\left|1+\frac{1}{z}\right| \le 1$

$$\left|1+\frac{1}{z}\right| \le 1$$

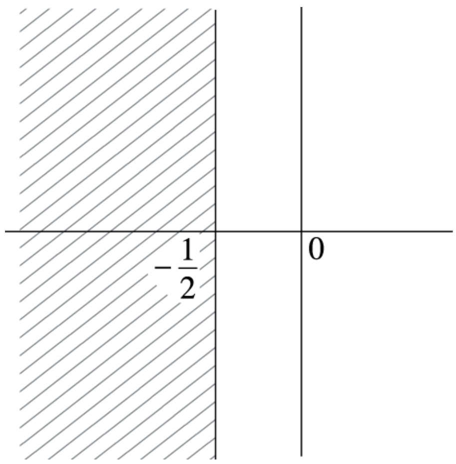
$$\Leftrightarrow \quad \left|1+z\right| \le \left|z\right|$$

$$\Leftrightarrow \quad \left|1+z\right|^2 \le \left|z\right|^2$$

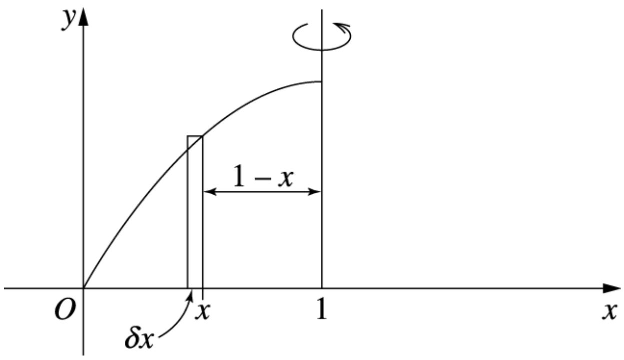
$$\Leftrightarrow \quad 1 + 2\operatorname{Re} z + \left|z\right|^2 \le \left|z\right|$$

$$\Leftrightarrow \quad 2\operatorname{Re} z \le -1$$

$$\Leftrightarrow \quad \operatorname{Re} z \le -\frac{1}{2}$$



Question 7 (a)



The approximate volume of a typical cylindrical shell is

$$2\pi(1-x)f(x)\,\delta x$$

since $2\pi(1-x)\delta x$ is the approximate area of the base and $f(x)$ the height.

Summing over the shells and letting $\delta x \to 0$.

$$V = 2\pi\int_0^1 (1-x)f(x) = 2\pi\int_0^1 (1-x)\frac{x}{1+x^2}\,dx$$

To compute the integral write

$$(1-x)\frac{x}{1+x^2} = \frac{x-x^2}{1+x^2} = \frac{x-\left(1+x^2\right)+1}{1+x^2}$$

$$= \frac{x}{1+x^2} - 1 + \frac{1}{1+x^2}$$

hence

$$V = 2\pi\int_0^1 \frac{x}{1+x^2} - 1 + \frac{1}{1+x^2}\,dx$$

$$= 2\pi\left[\frac{1}{2}\ln\left(1+x^2\right) - x + \tan^{-1}x\right]$$

$$= 2\pi\left[\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 - 1 + \tan^{-1}1 - \tan^{-1}0\right]$$

$$= 2\pi\left(\frac{1}{2}\ln 2 - 1 + \frac{\pi}{4}\right)$$

Question 7 (b) (i)

$$I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(x-4)}\,dx \quad = \quad \int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{(4-u)(-u)} - du$$

$$u = 4 - x$$
$$x = 4 - u$$
$$dx = -du$$

also $\cos^3\left(\frac{\pi}{8}(4-u)\right) = \cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}u\right)$

$= \sin^2\left(\frac{\pi}{8}u\right)$ (complementary identity)

thus $$I = \int_1^3 \frac{\sin^2 \frac{\pi}{8}u}{u(u-4)}\,du$$

$$= \int_1^3 \frac{\sin^2 \frac{\pi}{8}x}{x(x-4)}\,dx$$ (relabelling u as x)

Question 7 (b) (ii)

Hence

$$
\begin{aligned}
2I &= \int_1^3 \frac{\cos^2 \frac{\pi}{8}x}{x(x-4)}\,dx + \int_1^3 \frac{\sin^2 \frac{\pi}{8}x}{x(x-4)}\,dx \\
&= \int_1^3 \frac{\cos^2 \frac{\pi}{8}x + \sin^2 \frac{\pi}{8}x}{x(x-4)}\,dx \\
&= \int_1^3 \frac{dx}{x(x-4)}\,dx \\
&= -\frac{1}{4}\int_1^3 \frac{1}{x} + \frac{1}{4-x}\,dx \\
&= -\frac{1}{4}\left[\ln\left(\frac{x}{4-x}\right)\right]_1^3 \\
&= -\frac{1}{4}\left(\ln 3 - \ln\frac{1}{3}\right) \\
&= -\frac{1}{2}\ln 3
\end{aligned}
$$

Question 7 (c) (i)

Intersecting $y = mx + c$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$;

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$\left(b^2 + m^2a^2\right)x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

Now ℓ is a tangent, hence the above quadratic must have a double root. That is,

$$\triangle = \left(2a^2mc\right)^2 - 4\left(b^2 + m^2a^2\right)a^2\left(c^2 - b^2\right) = 0$$

$$\begin{aligned} a^2m^2c^2 &= \left(b^2 + m^2a^2\right)\left(c^2 - b^2\right) \\ \cancel{a^2m^2c^2} &= b^2c^2 - b^4 + \cancel{m^2a^2c^2} - m^2a^2b^2 \\ c^2 - b^2 - m^2a^2 &= 0 \qquad\qquad \text{(dividing by } b^2 \neq 0\text{)} \end{aligned}$$

Hence $\triangle = 0$ and we have a tangent.

Question 7 (c) (ii)

By the perpendicular distance formula, the distance from $S'(ae, 0)$ to $\ell : -y + mx + c = 0$ is

$$QS = \frac{|mae + c|}{\sqrt{1+m^2}}$$

Question 7 (a) (iii)

$$QS \times Q'S' = \frac{\left|(mae+c)(mae-c)\right|}{1+m^2}$$

$$= \frac{\left|m^2a^2e^2 - c^2\right|}{1+m^2}$$

$$= \frac{\left|m^2a^2e^2 - a^2m^2 - b^2\right|}{1+m^2} \qquad \text{using part (i)}$$

$$= \frac{\left|m^2a^2\left(1-\frac{b^2}{a^2}\right) - a^2m^2 - b^2\right|}{1+m^2} \qquad \text{since} \quad e^2 = 1 - \frac{b^2}{a^2}$$

$$= \frac{\left|\cancel{m^2a^2} - m^2b^2 - \cancel{a^2m^2} - b^2\right|}{1+m^2}$$

$$= b^2$$

Question 8 (a)

$$I_m = \int_0^1 x^m \left(x^2 - 1\right)^5 dx$$

$$= \frac{1}{2}\int_0^1 x^{m-1} \times 2x\left(x^2 - 1\right)^5 dx$$

$$= \left[\frac{1}{2}x^{m-1}\frac{1}{6}\left(x^2 - 1\right)^6\right]_0^1 - \frac{1}{2}\int_0^1 (m-1)x^{m-2} \cdot \frac{1}{6}\left(x^2 - 1\right)^6 dx$$

$$= 0 - \frac{(m-1)}{12}\int_0^1 x^{m-2}\left(x^2 - 1\right)^6 dx$$

$(m \geq 2)$

$$= -\frac{m-1}{12}\int_0^1 x^{m-2}\left(x^2 - 1\right)\left(x^2 - 1\right)^5 dx$$

$$= -\frac{(m-1)}{12}\int_0^1 x^m\left(x^2 - 1\right)^5 - x^{m-2}\left(x^2 - 1\right)^5 dx$$

$$= -\frac{(m-1)}{12}\left(I_m - I_{m-2}\right)$$

$$12I_m + (m-1)I_m = (m-1)I_{m-2}$$

$$I_m(m+11) = (m-1)I_{m-2}$$

$$I_m = \frac{(m-1)}{(m+11)}I_{m-2}$$

Question 8 (b) (i)

The number of ways in which 7 balls can be removed from the bag is 7^7.

The number of ways in which each ball is selected only once is $7!$. Therefore the probability that each ball is chosen once is $\frac{7!}{7^7} = \frac{6!}{7^6}$.

Question 8 (b) (ii)

This is $1 -$ (probability that each ball is selected)

which is $1 - \frac{6!}{7^6}$ (using part (i)).

Question 8 (b) (iii)

One of the 7 balls is not selected. This ball could be any one of the 7.

Of the 6 that are selected, one must be selected twice. This ball could be any of these 6 balls.

The ball selected twice could be selected in $\binom{7}{2}$ ways.

The remaining 5 balls can be selected in $5!$ ways.

So the number of ways of avoiding one ball is $7 \times 6 \times \binom{7}{2} \times 5!$.

The probability of this occurring is

$$\frac{7 \times 6 \times \frac{7!}{2}}{7^7} = \frac{3 \times 6!}{7^5}.$$

Question 8 (c) (i)

Since β is a root,

$$\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 = 0$$

$$\beta^n = -a_{n-1}\beta^{n-1} - \cdots - a_1\beta - a_0$$

$$\begin{aligned} |\beta|^n &= \left|a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0\right| \\ &\le |a_{n-1}||\beta|^{n-1} + \cdots + |a_1||\beta| + |a_0| \\ &\le M\left(|\beta|^{n-1} + \cdots + |\beta| + 1\right) \qquad \leftarrow \quad \text{a G.P.} \end{aligned}$$

Question 8 (c) (ii)

Hence $|\beta|^n \le M\dfrac{\left(|\beta|^n - 1\right)}{|\beta| - 1}$ (using result from part (i))

If $|\beta| > 1$ then

$$\begin{aligned} |\beta|^n\left(|\beta| - 1\right) &\le M\left(|\beta|^n - 1\right) \\ &< M|\beta|^n \\ |\beta| - 1 &< M \qquad \text{(divide by } |\beta|^n\text{)} \\ |\beta| &< 1 + M \end{aligned}$$

If $|\beta| \le 1$ then $|\beta| \le 1 + M$ is obvious

Question 8 (d)

Consider the polynomial $P(z) = \sum \left(\frac{c_k}{c_n} \right) z^k$

Where $z = x + \frac{1}{x}$

then any root β of $P(z)$ satisfies $|\beta| < 1 + M$

where $M =$ max value of $\left| \frac{c_0}{c_n} \right|, \left| \frac{c_1}{c_n} \right|, \cdots \frac{|c_{n-1}|}{|c_n|}$

notice $M \leq 1$ since $\left| \frac{c_k}{c_n} \right| \leq 1$

hence $|\beta| < 2$ (by part (c))

however, $\left| x + \frac{1}{x} \right| = |x| + \frac{1}{|x|}$ (if x is real)

≥ 2 (a standard 4 unit inequality)

Therefore $P(x) = 0$, hence $S(x) = 0$ has no real solution.

Notice that $x = 0$ is not a root of $S(x)$.