BOARD OF STUDIES
NEW SOUTH WALES

2012 HSC Mathematics Extension 1 'Sample Answers'

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This material is also provided to the Supervisor of Marking, to give some guidance about the nature and scope of the responses the committee expected students would produce. How sample answers are used at marking centres varies. Sample answers may be used extensively and even modified at the marking centre OR they may be considered only briefly at the beginning of marking. In a few cases, the sample answers may not be used at all at marking.

The Board publishes this information to assist in understanding how the marking guidelines were implemented.

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Section II

Question 11 (a)

$$\int_0^3 \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{1}{3}\tan^{-1}1 - \frac{1}{3}\tan^{-1}0$$

$$= \frac{1}{3}\times\frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Question 11 (b)

By the product rule

$$\frac{d}{dx}\left(x^2\tan x\right) = 2x\tan x + x^2\sec^2 x$$

Question 11 (c)

If $x > 3$, then $x < 2(x-3)$ ie $x < 2x - 6$

$6 < x$ (which satisifies $x > 3$)

If $x < 3$, then $x > 2(x-3)$ ie $x > 2x - 6$

$6 > x$ but $x < 3$

$\therefore\ x < 3$

$\therefore\ x > 6$ or $x < 3$

Question 11 (d)

$u = 2 - x, \quad x = 2 - u$

$du = -dx \quad dx = -du$

If $x = 1$, then $u = 1$ and if $x = 2$, then $u = 0$

Hence

$$\int_1^2 x(2-x)^5\,dx = -\int_1^0 (2-u)u^5\,du$$

$$= \int_0^1 (2-u)u^5\,du$$

$$= \int_0^1 2u^5 - u^6\,du$$

$$= \left[\frac{u^6}{3} - \frac{u^7}{7}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{7}$$

$$= \frac{4}{21}$$

Question 11 (e)

$$\binom{8}{3}\binom{10}{4}$$

Question 11 (f) (i)

We need $\binom{12}{k}\left(2x^3\right)^k\left(-\frac{1}{x}\right)^{12-k}$ = constant

This means $x^{3k}\left(x^{-1}\right)^{12-k} = x^0$

$$= 1$$

$$\therefore\ 3k - (12 - k) = 0$$

$$4k - 12 = 0$$

$$k = 3$$

Hence the constant term is

$$\binom{12}{3}\left(2^3\right)(-1)^9 = -8\binom{12}{3}$$

$$= -1760$$

Question 11 (f) (ii)

$4k - n = 0,$ so n must be a multiple of 4

Question 12 (a)

If $n = 1$, then $2^3 - 3 = 8 - 3$

$= 5$ which is divisible by 5.

Asssume true for $n = k$

ie $2^{3k} - 3^k = 5j$ for some integer j

Then if $n = k + 1$

$$2^{3(k+1)} - 3^{k+1} = 2^{3k+3} - 3^k \times 3$$

$$= 8\left(5j + 3^k\right) - 3 \times 3^k \quad \text{(using the assumption)}$$

$$= 8 \times 5j + 5 \times 3^k$$

$$= 5\left(8j + 3^k\right) \quad \text{which is divisible by 5.}$$

Hence the claim is true for $n = k + 1$. Since shown true for $n = 1$, so is true for $n = 2, 3, \ldots$ and so true for all integers $n \ge 1$.

Question 12 (b) (i)

We need $4x - 3 \ge 0$, so the domain is $x \ge \frac{3}{4}$

Question 12 (b) (ii)

$$y = \sqrt{4x - 3}$$

$$y^2 = 4x - 3$$

$$4x = y^2 + 3$$

$$x = \frac{y^2 + 3}{4} \quad \text{for } y \ge 0$$

Hence $f^{-1}(x) = \frac{x^2 + 3}{4}$ for $x \ge 0$

Question 12 (b) (iii)

If $x = y$, then

$$f(x) = \sqrt{4x - 3} = x$$

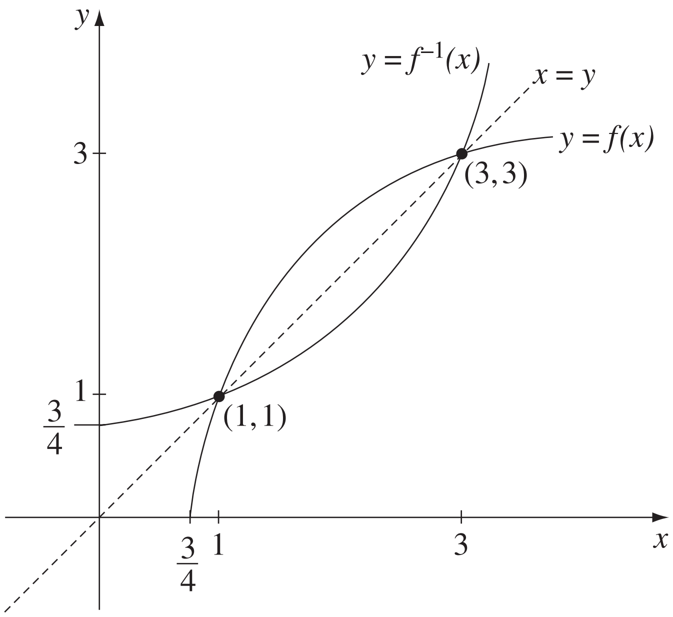
Hence $x^2 = 4x - 3$

ie $x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

The points of intersection therefore are $(1, 1)$ and $(3, 3)$

Question 12 (b) (iv)



Question 12 (c) (i)

The probability that Kim wins is $\frac{2}{5}$

Question 12 (c) (ii)

The probability that Kim wins exactly three games is

$$\binom{6}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^3$$

(binomial probability)

Question 12 (d) (i)

Using Pythagoras' theorem on the right angled triangles *ABC*, *OCB* and *OCA*

$$(y+k)^2 = (t^2+y^2)+(t^2+k^2)$$
$$y^2+2ky+k^2 = 2t^2+y^2+k^2$$
$$ky = t^2$$
$$y = \frac{t^2}{k}$$

Hence $x = t$ and $y = \frac{t^2}{k}$ are the coordinates of P.

Question 12 (d) (ii)

The vertex is at $(0, 0)$ and $y = \frac{1}{k}x^2$

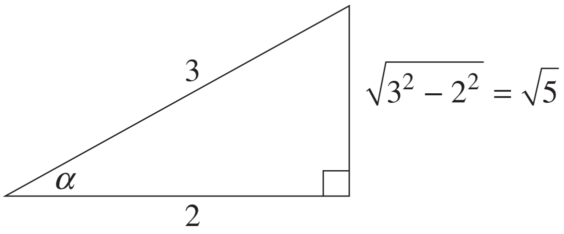
$$\therefore\ 4a = k$$
$$a = \frac{k}{4}$$

so the focus is $\left(0, \frac{k}{4}\right)$.

Question 13 (a)

If we set $\alpha = \cos^{-1}\frac{2}{3}$, then

$$\sin\left(2\cos^{-1}\frac{2}{3}\right) = \sin 2\alpha$$
$$= 2\sin\alpha\cos\alpha$$



From the diagram $\sin\alpha = \frac{\sqrt{5}}{3}$ and $\cos\alpha = \frac{2}{3}$,

so

$$\sin\left(2\cos^{-1}\frac{2}{3}\right) = 2 \times \frac{\sqrt{5}}{3} \times \frac{2}{3}$$
$$= \frac{4}{9}\sqrt{5}$$

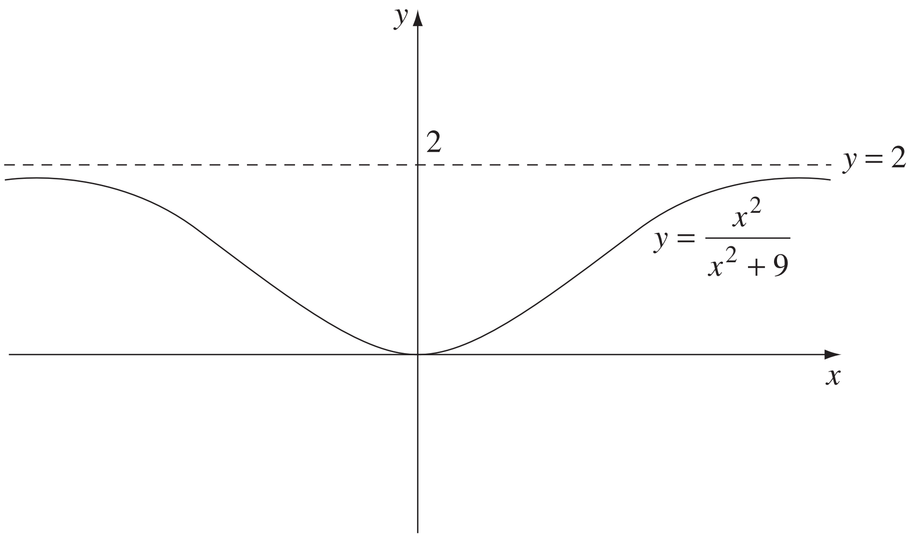
Question 13 (b) (i)

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2+9} = \lim_{x\to\infty} \frac{2}{1+\frac{9}{x^2}}$$
$$= \frac{2}{1}$$
$$= 2$$

Hence the horizontal asymptote is $y = 2$

Question 13 (b) (ii)

When $x = 0$, $y = 0$. Also $y \geq 0$ and it is an even function



Question 13 (c) (i)

$\dot{x} = -12\sin 2t + 16\cos 2t$

$\ddot{x} = -24\cos 2t - 32\sin 2t$

$= -4(6\cos 2t + 8\sin 2t)$

$= -4(5 + 6\cos 2t + 8\sin 2t - 5)$

$= -4(x - 5)$

$= -2^2(x - 5)$

Question 13 (c) (ii)

Displacement zero means $x = 0$

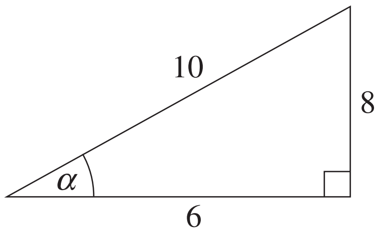
$$5 + 6\cos 2t + 8\sin 2t = 0$$

$$6\cos 2t + 8\sin 2t = -5$$

Rewrite as $10\cos(2t - \alpha) = -5$

$$10(\cos 2t \cos\alpha + \sin 2t \sin\alpha) = -5$$

Looking at the triangle



we see that

$$\tan x = \frac{8}{6}$$

$$x = \tan^{-1}\frac{8}{6}$$

$$x \approx 0.9273$$

Hence $\cos(2t - 0.9273) \approx -\frac{5}{10} = -0.5$

$$2t - 0.9273 = \frac{2\pi}{3} \quad \left(\text{since } \cos\left(\frac{2\pi}{3}\right) = -0.5\right)$$

$$= 2.0944$$

$$2t = 3.02169$$

$$t = 1.511$$

Question 13 (d) (i)

$$\frac{dC}{dt} = 1.4e^{-0.2t} + 1.4(-0.2)te^{-0.2t}$$

$$= 1.4(1 - 0.2t)e^{-0.2t}$$

Hence $\frac{dC}{dt} = 0$, if $1 - 0.2t = 0$, so $t = 5$

Now

$$\frac{d^2C}{dt^2} = -1.4(0.2)e^{-0.2t} - 1.4(0.2)(1 - 0.2t)e^{-0.2t}$$

$$= -1.4(0.2)(2 - 0.2t)e^{-0.2t}$$

At $t = 5$,

$$\frac{d^2C}{dt} = -1.4(0.2)(2 - 1)e^{-1} < 0$$

Hence the maximum occurs at $t = 5$ hours.

Question 13 (d) (ii)

When $C = 0.3$

$$0.3 = 1.4te^{-0.2t}$$

$$\therefore f(t) = 1.4te^{-0.2t} - 0.3$$

$$f'(t) = 1.4t(-0.2)e^{-0.2t} + e^{-0.2t}(1.4)$$

$$= 1.4e^{-0.2t}(-0.2t + 1)$$

$$\therefore f(20) = 1.4(20)e^{-0.2(20)} - 0.3$$

$$= 28e^{-4} - 0.3$$

and $f'(20) = 1.4e^{-0.2(20)}(-0.2 \times 20 + 1)$

$$= 1.4e^{-4}(-4 + 1)$$

$$= -4.2e^{-4}$$

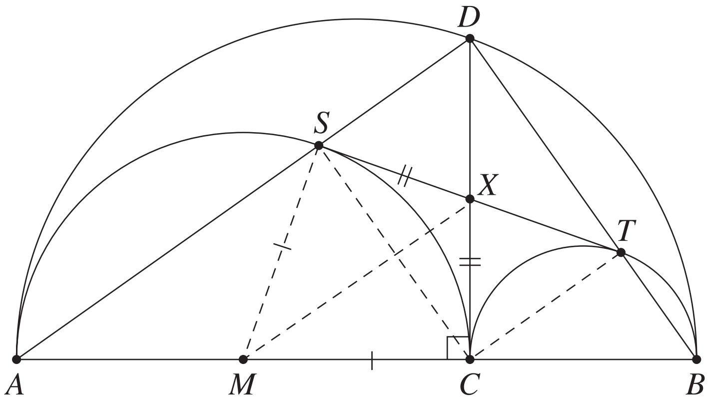
now, a better approximation is given by:

$$t = 20 - \frac{f(20)}{f'(20)}$$

$$= 20 - \frac{0.21283\ldots}{-0.0769\ldots}$$

$= 22.766\ldots$ ie after approximately 22.8 h

Question 14 (a) (i)



$\angle ASC = 90^\circ$ (angle in semi-circle, diameter AC)

$\angle DSC = 90^\circ$ (adjacent supplementary angles)

Similarly $\angle DTC = 90^\circ$

$\angle ADB = 90^\circ$ (angle in semi-circle, diameter AB)

Hence $CTDS$ is a rectangle since all angles are 90°

Question 14 (a) (ii)

As the diagonals of a rectangle are equal and intersect at their midpoints, $XS = XC$

As S and C are on the same circle with centre M, $MS = MC$ (equal radii)

MX is common to the triangles MXS and MXC

Hence $\triangle MXS \equiv \triangle MXC \quad (SSS)$

Question 14 (a) (iii)

$\angle MCX = 90^\circ$ as $DC \perp AB$ (given)

$\angle MCX = \angle MSX$ (corresponding angles in congruent triangles)

ie $\angle MSX = 90^\circ$

Hence radius $MS \perp ST$ at the point of contact, and ST is a tangent.

Question 14 (b) (i)

The maximum height occurs when $\dot{y} = 0$

$$\dot{y} = 70\sin\theta - 9.8t$$

$$= 0$$

$$9.8t = 70\sin\theta$$

$$t = \frac{70\sin\theta}{9.8}$$

$$\therefore \; y = 70 \times \frac{70}{9.8}\sin^2\theta - \frac{4.9 \times 70 \times 70\sin^2 0}{9.8 \times 9.8}$$

$$= 500\sin^2\theta - 250\sin^2\theta$$

$$= 250\sin^2\theta$$

Question 14 (b) (ii)

$$t = \frac{70\sin\theta}{9.8}$$

$$\therefore x = 70 \times \frac{70\sin\theta}{9.8}\cos\theta$$

$$= 500\cos\theta\sin\theta$$

$$= 250(2\sin\theta\cos\theta)$$

$$= 250\sin 2\theta$$

Question 14 (b) (iii)

We want $125 \le 250\sin 2\theta \le 180$

$$\frac{1}{2} \le \sin 2\theta \le \frac{18}{25}$$

$\therefore \quad 30^\circ \le 2\theta \le 46.05^\circ \quad$ or $\quad 133.95^\circ \le 2\theta \le 150^\circ$

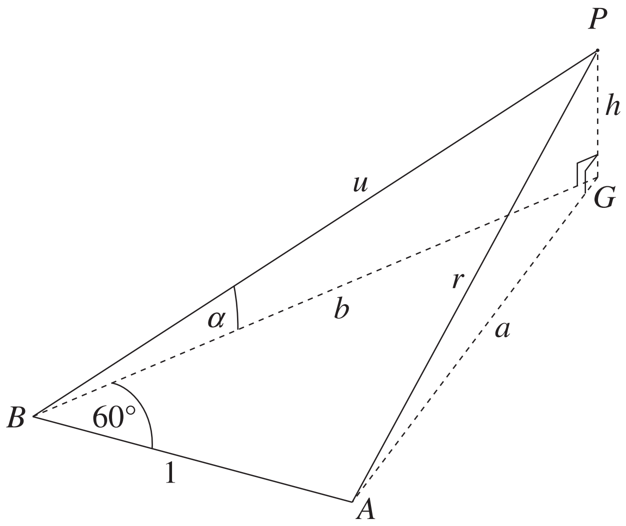
$15^\circ \le \theta \le 23.05 \quad$ or $\quad 67.98^\circ \le \theta \le 75^\circ$

when $\theta = 23.05^\circ \quad$ max ht $= 38.32$ m $\quad < 150$ m

when $\theta = 67.98^\circ \quad$ max ht $= 214.86$ m $\quad > 150$ m

$\therefore \; 67.98^\circ \le \theta \le 75^\circ$

Question 14 (c) (i)



Let $h = PG$

$a = AG$

$= \sqrt{r^2 - h^2}$

$b = BG$

$= u\cos\alpha$

$= \sqrt{u^2 - h^2}$

By the cosine rule on the triangle ABG

$$a^2 = 1^2 + b^2 - 2 \times 1 \times b\cos 60^\circ$$

$$r^2 - h^2 = 1 + \left(u^2 - h^2\right) - 2u\cos\alpha \times \frac{1}{2}$$

$$r^2 = 1 + u^2 - u\cos\alpha$$

$$r = \sqrt{1 + u^2 - u\cos\alpha}$$

Question 14 (c) (ii)

$v = \dfrac{du}{dt}$, so

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt}$$

$$= \frac{(2u - \cos\alpha)v}{2\sqrt{1 + u^2 - u\cos\alpha}}$$

Five minutes after take off $u = vt = 360 \times \dfrac{1}{12} = 30$

Hence

$$\frac{dr}{dt} = \frac{360(60 - \cos\alpha)}{2\sqrt{1 + 30^2 - 30\cos\alpha}}$$

$$= \frac{180(60 - \cos\alpha)}{\sqrt{901 - 30\cos\alpha}} \text{ km/h}$$